1982

On integrating spatial and temporal considerations into the economic theory of clubs

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8224233

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ON INTEGRATING SPATIAL AND TEMPORAL CONSIDERATIONS INTO THE ECONOMIC THEORY OF CLUBS

Iowa State University PH.D. 1982

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

On integrating spatial and temporal considerations into the economic theory of clubs

by

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A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major: Economics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1982

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PREFACE

"Timing is all important."

- Ernesto Javier

CHAPTER I. INTRODUCTION

The economic theory of clubs is aimed at filling in the "awesome Samuelson gap" [5, p. 1] between the existing theories of purely public and purely private goods. It should be recalled that a purely public good is one that is completely non-rival in consumption. If additional people consume a fixed quantity of a public good, the utility of those already consuming the good is unaffected. On the other hand, private goods are completely rival in consumption. If one person consumes a given quantity of a private good, that amount is simply unavailable to anyone else. However, there is a group of goods and services which fall into the gap between these categories. These are goods that can be shared, unlike private goods, but not endlessly so, as with public goods. Such items are somewhere between the two extremes, and are known as impurely public or club goods.

A common example of a club good is a swimming pool. Unlike a purely private good, more than one person can use the pool at the same time. However, as more and more people get into the pool, the enjoyment of those already in it diminishes. In a word, the pool gets conjested.

This situation creates a classic economic problem. Because the pool can be shared, per-person cost can be reduced if a club[1] is formed and the facility is shared. The more people there are in the club, the

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[1] It should be noted that clubs may be either public or private. Individuals may join together to form a club, or may create a governmental unit for the same purpose. However, this dissertation implicitly assumes that the clubs discussed are public.

lower the per-person cost, but the more congested the pool will be. The "right" number of people for the club depends upon the members' evaluation of the gains to be made from lowering per-person cost and the losses to be sustained from additional congestion. It is entirely possible that an individual will be rich enough and sufficiently averse to congestion that he will buy his own pool. Note that the swimming pool is still a club good, despite the fact that it is owned by a single person. Property rights do not alter the intrinsic nature of goods. Thus, many goods generally considered private goods (such as televisions and automobiles) are by nature actually club goods.

Congestion manifests itself in the swimming pool when the addition of extra swimmers inhibits one's ability to swim about freely. However, congestion takes on different forms with different types of club goods. A congested tennis court, for example, is not one where there is not enough room to swing a racket, but rather where there are more people wanting to play than for whom there is room. The result is the creation of a waiting line or "queue", and the congestion costs manifest themselves in terms of waiting time costs. Likewise, a congested fire suppression system is one where there are more fires than units available to put them out. Once again, congestion costs appear in terms of time: in this case, time spent waiting for a fire-fighting unit to become available.

It should be obvious that, all else equal, increasing the number of people in the tennis club or fire suppression system will increase the amount of time one might expect to wait for an open court or a free fire truck. These time costs must therefore be weighed against the reductions in per-person cost of providing the club good that stem from increasing club membership. But increasing club membership has one additional implication: it increases the physical size of the club's "district". Fire trucks will spend more time in transit between fires in larger districts, ceteris paribus, and tennis players will spend more time traveling to and from the courts. Additional space thereby translates into additional time.

Surprisingly, the formal economic theory of clubs has so far virtually ignored these important spatial and temporal considerations [31]. Some work has been done along these lines, but almost exclusively by operations research practitioners [4, 6, 7, 10, 20, 21, 22, 23]. Thus, no effort has been made to incorporate this work into a more general theoretical framework such as the economic theory of clubs.

The primary purpose of this dissertation is to take a step towards eliminating this deficiency by integrating spatial and temporal considerations into the economic theory of clubs. In so doing, certain techniques developed by the operations research discipline have been used, with the secondary purpose of making economists aware of the work being done in that area. In addition, this dissertation incorporates a Von Neuman-Morganstern approach to expected utility maximization. This idea also has never been used in conjunction with formal club theory [31].

Chapters II and III of this work focus on these issues as they apply to local emergency services, with a special emphasis on fire suppression systems. Chapter IV extends the analysis to include man-made recreational facilities, with tennis courts being used as a specific example.

CHAPTER II. FIRE SUPPRESSION DISTRICTS: A GENERAL MODEL

The Elements of the Problem

In an emergency, time is of the essence. The longer a fire burns, for example, the more damage it will do. The goal of every fire suppression service is, therefore, to minimize the length of time a fire burns, given the ever-present cost constraint.

The total time that a fire burns depends upon four things: detection and reporting time, queuing time, travel time, and service time. Detection and reporting time is the time that elapses between the moment the fire starts and the moment it is reported to the fire station. To a large extent, this variable cannot be influenced by changes in the configuration of the fire district (i.e. by having more fire companies[1] or fewer people in the "club"). Fire detection and reporting is for the most part the responsibility of the individual, who may choose to install smoke detectors or similar devices. Hence, detection and reporting time is a variable assumed to be exogenous to the model, since it is beyond the control of the group (with the possible exception of building code legislation, but that is beyond the scope of this work).

Queuing time is the time that elapses between the moment the fire station receives the report of a fire and the moment a fire company

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[1]The phrase "fire company" will be used to refer to a standard fire-fighting unit. Surprisingly, there has been very little research done on the nuts and bolts issues of how many fire-fighters to put into each company, the types of trucks to use, etc. [33]. Hence, the exact composition of each fire company is assumed to be exogenously determined.

becomes available to respond to the report. With the assumption of autonomous fire districts, if all fire companies are already busy, the caller must wait until one finishes its current task before one can respond to his alarm. There is a branch of operations research devoted to the study of queues, and Appendix A of this paper offers the reader an introduction to the area. As shall become apparent later on, changing the configuration of the fire district can significantly affect queuing time.

Travel time is the time that elapses between the moment a fire company is dispatched to a fire and the moment it arrives on the scene. All else equal, the more people there are in the fire district, the larger the physical size of the district, and hence, the greater average or median travel time will be.

Service time is the amount of time that elapses between the moment the fire company arrives at the scene of the fire and the moment the fire is finally extinguished. Service time depends upon a variety of factors including the training of the fire company, its size, the technology it possesses, and the nature of the particular problem at hand. Since the composition of a fire company has been assumed to be exogenously determined, it follows that service time is also an exogenous variable. Note, however, that both service time and travel time indirectly influence queuing time, since the quicker a unit can get to and extinguish a fire, the sooner it will be available for another assignment.

The Decision-Making Framework

As indicated above, the two variables that can be affected by altering the configuration of fire districts are queuing time and travel time. Queuing time, once determined, will be the same for everyone in the district. The origin of the call for help in no way affects the probability that the fire companies will all be busy. However, travel time depends upon where the individual lives relative to the fire station. Someone living adjacent to the fire station can expect travel time to be much less than it would be for someone living a few miles away. This observation is important when considering how different individuals perceive the "efficiency" of a particular district configuration.

One issue to be decided is the level of per-person cost (or "dues" in club theory jargon). With the assumption of equal cost sharing, determining the level of dues is equivalent to determining the ratio of fire companies (S) to people in the district (N). This is because total cost is equal to the (annualized) cost of each fire company (C) times the number of companies. Hence, cost per-person is $(C \cdot S)/N$. With C given and independent of S, there is a one-to-one correspondence between dues and S/N.

But determining S/N is only part of the story. For a given S/N, the levels of S and N must still be decided. If, for example, $S/N = 1/10{,}000$, one must determine whether the district should contain 1 fire company and 10,000 people, 2 companies and 20,000 people, or 10 companies and 100,000 people. Obviously, including more people in the district will increase travel time to the people living at the outer edge of the district. However, increasing S and N proportionally will reduce mean queuing time. This economy of scale in queuing time is "a property which is common to nearly all realistic queuing models" [6, p. 18]. An interesting political problem thus arises because people living near the fire station will want S and N to increase indefinitely (given S/N), since it will reduce queuing time without affecting travel time to them. However, as N increases, people living on the boundary will expect large travel times, and will therefore desire smaller districts. The equilibrium sized district (again, given S/N) postulated in this work is where the (expected) utility of the individuals living the median distance from the center of the district is greater than the (expected) utility of individuals living the median distance from the center of any other sized district. (Appendix B shows how median distance is calculated. It also demonstrates that median distance is larger than mean distance. Thus, the result obtained here is certainly different from that which would occur under a "veil of ignorance". Under such a veil, "no one knows his place in society" [28, p. 12], so no one would have any idea where the fire stations would be located. If everyone were risk neutral, they would vote such that the district were configured so as to maximize the expected utility of the individual living the mean distance from the center of the district. If risk aversion were prevalent, district size would depend upon the degree of risk aversion, and would coincide with the result obtained here only by accident.) The equilibrium is also sensitive to agenda control and other public choice issues beyond the scope of this dissertation.

Having determined what S and N will be for each possible S/N, every individual will know his position relative to the fire station for every S/N, and will vote so as to maximize his own expected utility. By invoking the median voter theorem in a majority-rules voting framework, it can be argued that the median preference will dominate [24].

The Nature of the Utility Function

The model presented here defines utility as being a function of wealth (Z). Mathematically, $U = U(Z)$. The amount of wealth an individual possesses depends upon his initial endowment, payments made for services to protect that endowment (dues), and potential loss due to fire.

Since the need for an emergency service such as a fire suppression company is uncertain, behavior consistent with the Von Neuman-Morganstern expected utility maximization hypothesis is postulated. The essence of this hypothesis is that under conditions of uncertainty, individuals will behave so as to maximize the expected value of utility. (Proofs of this contention, given the necessary assumptions, can be found in References 15, 18, and 39.)

In this chapter, no restrictions are placed on the form of the utility function. For illustrative purposes, a risk averse utility function will be employed (i.e. $U'(Z) > 0$, $U''(Z) < 0$, where $U'(Z)$ and $U''(Z)$ are the first and second derivatives of utility with respect to wealth), but the model is applicable to other utility functions as well. In the next chapter, a risk neutral utility function will be specified which will allow us to simplify the analysis.

Choosing S and N for Each Level of Per-Person Cost

As noted above, there is a one-to-one correspondence between the level of per-person cost (dues) and the ratio of fire companies to people (S/N). It was also noted that S and N were chosen for a particular S/N so as to make the expected utility of the individuals living the median distance from the center of the district greater than the expected utility of similarly placed individuals in alternatively sized districts. This section describes that process.

Figure 1 below displays two rays from the origin, labeled $(S/N)_1$ and $(S/N)_2$.

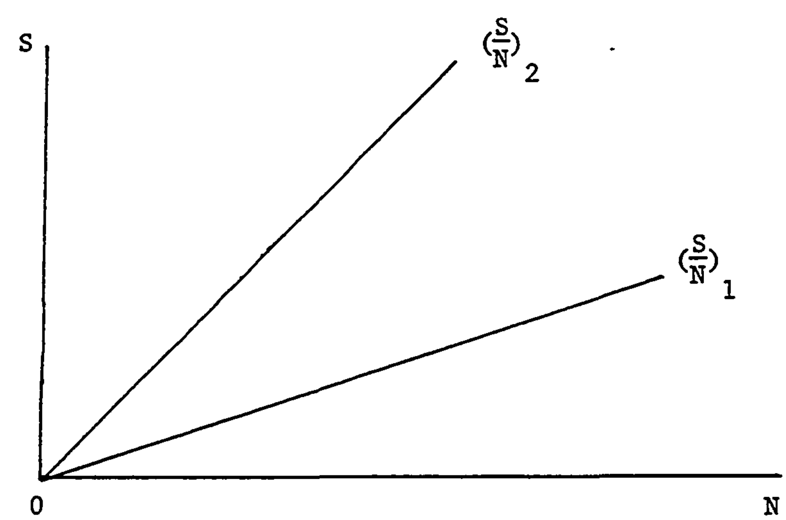

Figure 1. Iso-dues lines

Along each ray, S/N is constant. If the cost of each fire company is independent of the number hired, then along each ray per-person cost or dues is also constant. Since $(S/N)_2 > (S/N)_1$, dues for $(S/N)_2$ are greater than dues for $(S/N)_1$ [1, 2, 3]. The task is to choose a particular ray, and then to locate the appropriate S and N.

Z_N is defined as wealth net of dues. Therefore, $Z_N = Z$ - dues, and is the value of wealth if there is no fire. Z_a is defined as wealth that is not subject to fire damage (money in a savings account, for example). Hence, Z_a is the value of wealth if a catastrophic fire occurs which destroys all burnable wealth. $Z_N - Z_a$ is thus the amount of wealth subject to loss by fire.

It is hypothesized that loss due to fire (L) is a function of the length of time the fire burns (T). Mathematically, $L = L(T)$. The shape of this loss function depends upon a variety of factors including the construction of the building, the nature of its contents, the number of doors and windows, climatic conditions, and the value of the building and its contents. In this chapter, no restrictions are placed on the shape of the loss function, though for illustrative purposes it is drawn such that $L'(T) > 0$ and $L''(T) > 0$. In the next chapter, it will be assumed to be linear to simplify the analysis.

Let P_F be the probability that a fire will occur at any particular residence (assumed to be equal for everyone). Expected utility for any individual can then be expressed as[1]

$$E[U(Z)] = (1 - P_F)[U(Z_N)] + P_F \int_{Z_a}^{Z_N} U(Z)f(Z)dZ$$

---

[1]This formulation is standard, and can be found in Reference 35, for example.

where f(Z) is the probability density function of wealth in the event of a fire. This density function is derived from the probability density function of T and from the loss function L(T).

Since T is the sum of four random variables (detection and reporting time, queuing time, travel time, and service time), its density function is obtained from the density functions of its four-component random variables. (An example of the general technique is presented in Appendix C.)

The probability density function of T is also the probability density function of L, since P(T=t) = P(L(T) = L(t)). In other words, if P_1 is the probability that the fire will burn for a period of time, t, then P_1 is the probability that loss will be L(t).

Similarly, there is a one-to-one correspondence between loss and wealth remaining after the fire. If loss due to fire is L(t), then wealth must be Z_N - L(t). Therefore, if loss L(t) occurs with probability P_1, wealth Z_N - L(t) occurs with probability P_1. The probability density function of T is, therefore, transformed into the probability density function of wealth, f(Z). The idea is illustrated in Figure 2.

The diagram in the northwest quadrant shows the probability density function of T, and shows the probability of a fire burning for any particular length of time, t. The graph in the southwest corner shows the loss function, L(T). The probability of any particular amount of loss, L(t), can be seen by going directly north to the probability density function of T. Note that loss due to fire can be a maximum of Z_N - Z_a,

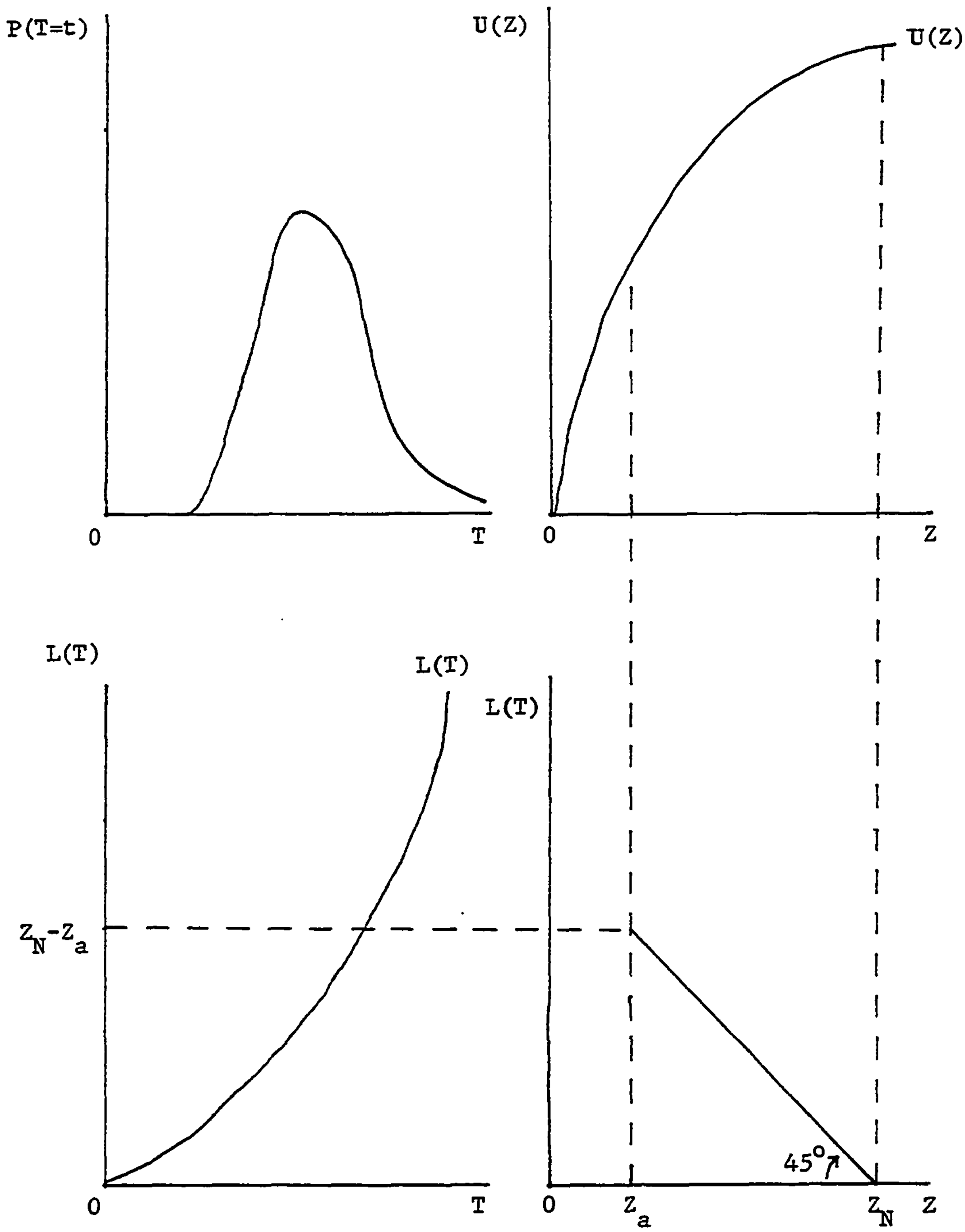

Figure 2. Deriving f(Z)

since by definition Z_a is wealth that will still exist even if the fire destroys all flammable wealth.

The picture in the southeast quadrant translates loss into remaining wealth. If loss is zero, then wealth is $Z_N - 0 = Z_N$. If loss is $Z_N - Z_a$, then wealth is $Z_N - (Z_N - Z_a) = Z_a$. Since one can determine the probability of loss being any particular amount, L(t), one can also determine the probability of wealth being any particular amount, Z_N - L(t). Finally, by moving up to the utility function in the northeast quadrant, one can determine the utility of any particular level of wealth.

The position of the utility function obviously depends upon the exact function specified. In the risk averse case pictured, the function is concave from below. A risk neutral utility function would be linear, and a risk loving utility function would be convex from below.

The location of Z_N in the southeast corner depends upon the individual's initial endowment of wealth, as well as dues (which are a function of C). The location of Z_a depends upon the fraction of Z_N that the individual had in a form not subject to fire damage.

The shape of the probability density function of T as depicted in Figure 2 is somewhat arbitrary, since its exact shape depends upon the density functions of the component variables (please see Appendix C). Note, however, that even with S/N fixed, the shape of this density function will change as S and N change. As previously indicated, increasing S and N (S/N constant) will cause the mean of the queuing time density function to decline, but the mean of the travel time density function will increase (a larger N implies a physically larger district). The S and N which ultimately result in the f(Z) which generates the greatest expected utility for the individuals living the median distance from the center of the district will be chosen. Designate these values as S^* and N^*.

Unfortunately, the complexity of the queuing time equation and the difficulty of calculating the probability density function of T (please see Appendices A and C) preclude an analytic solution to this problem. The purpose of Chapter III, therefore, is to examine the model under certain simplifying assumptions. These assumptions will still not allow for an exact analytic solution, but they will allow solutions to be obtained by means of simulations.

The Expansion Path and the Choice of S/N

By calculating S^* and N^* for each S/N, an entire set of equilibrium-sized districts is identified. This collection of points is known as an expansion path, and delineates the district configurations potentially available. Along an expansion path, a larger S/N implies both better service and higher dues. Off the expansion path, a larger S/N still implies higher dues, but not necessarily better service [2]. Note that the position of the expansion path depends upon all of the elements needed to choose each S^* and N^*, including the utility function.

Given the S^* and N^* that will prevail for each S/N, each voter will desire the S/N that will maximize his expected utility. In this situation, each person knows the travel time and queuing time that will be associated with each level of dues. A larger S/N will affect expected utility in two ways. First, the mean of the probability density function of T will be reduced. Secondly, dues will increase, so Z_N (and probably Z_a) will be reduced. Figure 3 compares two alternatives, $(S/N)_1$ and $(S/N)_2$, where $(S/N)_2 > (S/N)_1$.

By choosing $(S/N)_2$, the individual will have a higher probability of fire being extinguished sooner than if he were to choose $(S/N)_1$. However, his wealth in the event that there is no fire is reduced by the amount of extra dues from Z_{N_1} to Z_{N_2}. Mathematically, the individual must compare the value of expected utility with $(S/N)_2$ to that with $(S/N)_1$, i.e. compare

$$E[U(Z)]_2 = (1 - P_F)[U(Z_{N_2})] + P_F \int_{Z_{a_2}}^{Z_{N_2}} U(Z)f_2(Z)dZ$$

to

$$E[U(Z)]_1 = (1 - P_F)[U(Z_{N_1})] + P_F \int_{Z_{a_1}}^{Z_{N_1}} U(Z)f_1(Z)dZ \ .$$

Each individual will select the S/N that produces the largest $E[U(Z)]$. Obviously, different individuals will have different preferences. Wealth, utility functions, location (and hence, $f(Z)$) will all differ from person to person. In addition, the fraction of wealth kept "out of harm's way" (Z_a) will differ, as might detection and reporting time (due to individual efforts to install smoke alarms, etc.). The loss function will also vary from individual to individual, as it depends on a variety of personal factors described before. The individual with the median preference will dominate in the choice of S/N in a majority-rules voting framework. Note that this individual

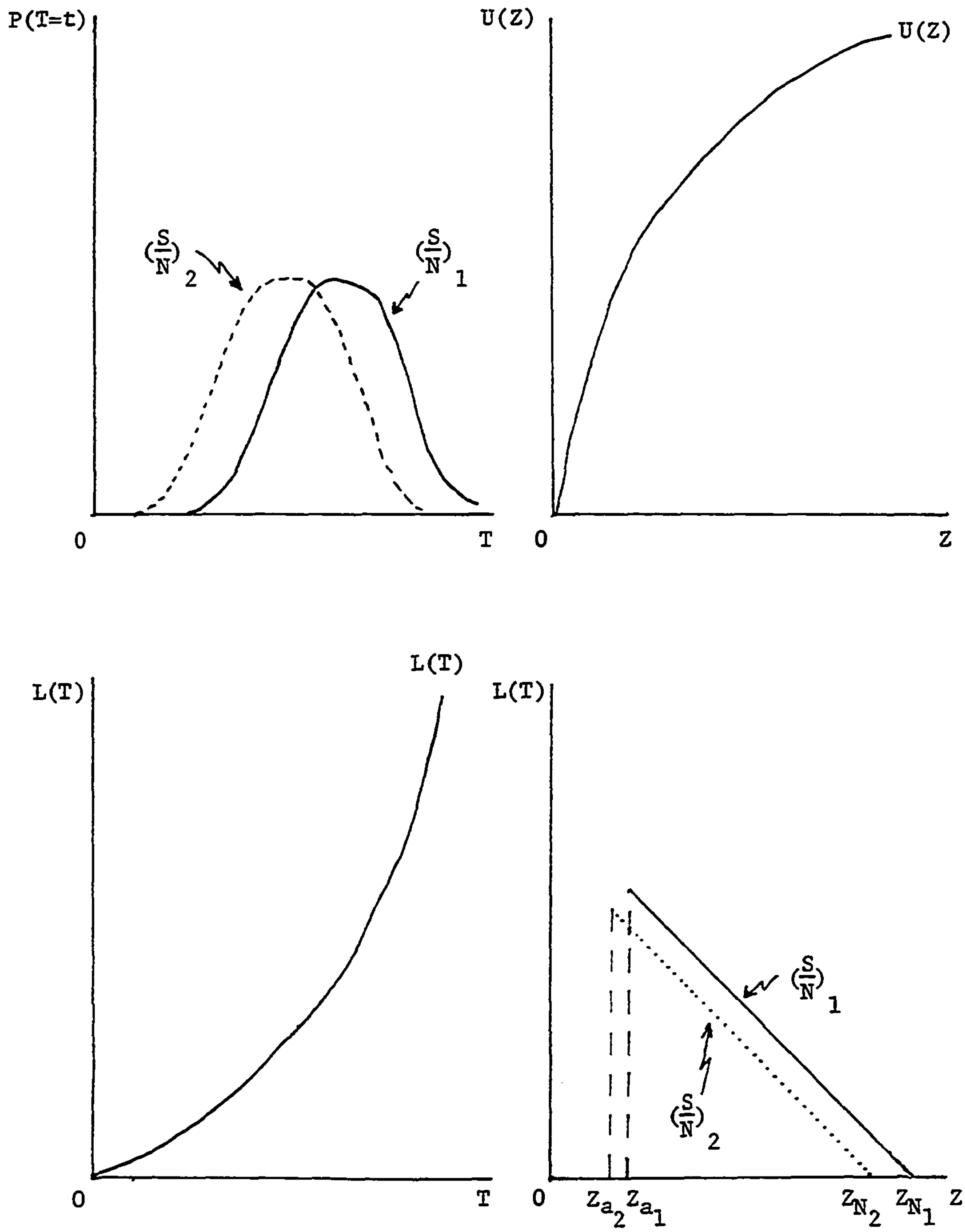

Figure 3. Two points on the expansion path compared

need not be among the individuals living the median distance from the center of any particular district. Individuals choose their expected utility maximizing S/N given the S^* and N^* associated with each S/N. With that information, they will know where they will live relative to the fire station for each S/N, and will vote on the basis of that information.

CHAPTER III. FIRE SUPPRESSION DISTRICTS: A RESTRICTED MODEL

The Nature and Purpose of the Restrictions

This chapter will employ the same model as presented in the previous chapter, but with three modifications. The first is the restriction that all utility functions be risk neutral. According to the Arrow-Pratt measure of absolute risk aversion [27], an individual is risk neutral if his utility function is such that:

$$\frac{-U''}{U'} = 0 \; .$$

Since the second derivative of any linear function is equal to zero, a linear utility function is a risk neutral utility function. One such utility function is $U = bZ$, where b is a positive constant, and Z is wealth as before. However, all Von Neuman-Morganstern utility functions are unaffected by linear transformations [18, p. 36], so the above function may be multiplied by $1/b$. The resulting function, $U = Z$, will be the function used here.

The second modification is to restrict the loss function to be linear in time. In other words, loss will be equal to $L = gT$ where g is a positive constant (dollars lost per unit of time) and T, as before, is the total length of time the fire burns. (Empirical work on the nature of this loss function has been sparse and inconclusive, and has focused mainly on response distance as the explanatory variable [20]. However, there is weak evidence that loss is in fact a linear function of response time [20].) The expected value of T is simply the sum of mean detection and reporting time, mean queuing time, mean travel time, and mean service time.

Risk neutral utility functions have the unique property of having the utility of the expected value of wealth equal to the expected value of the utility of wealth [18, p. 49]. Linear loss functions are such that the loss associated with the expected value of T is equal to the expected value of loss. In equation form,

$$E[U(Z)] = U[E(Z)]$$

and

$$E[L(T)] = L[E(T)] \ .$$

These two restrictions allow one to focus exclusively on the expected values of all variables. Thus, instead of having to employ whole density functions, only the means of density functions need be used. This simplification will make the analysis considerably more manageable.

The third restriction is the assumption that all individuals in a district have the same amount of wealth. This assumption will simplify a portion of the comparative statics of the model. If Tiebout [34] is to be believed, this assumption may be an approximation of what frequently occurs in the real world anyway.

Choosing S and N for Each Level of Per-Person Cost

As before, detection and reporting time as well as service time are assumed to be exogenous to the model. The concerns here, then, are the effects on the expected value of queuing time (W_q) and on the expected value of travel time to the median individuals (TT) of changing the district's configuration. In other words, how do W_q and TT change as S and N change, given S/N?

As mentioned in Chapter II, a characteristic of mean queuing time is that it declines as S and N increase proportionately. Doubling both the number of fire companies and the number of people in the district will cause W_q to decline. Of course, the larger S/N is, the smaller W_q is for any particular N. These ideas are illustrated in Figure 4.

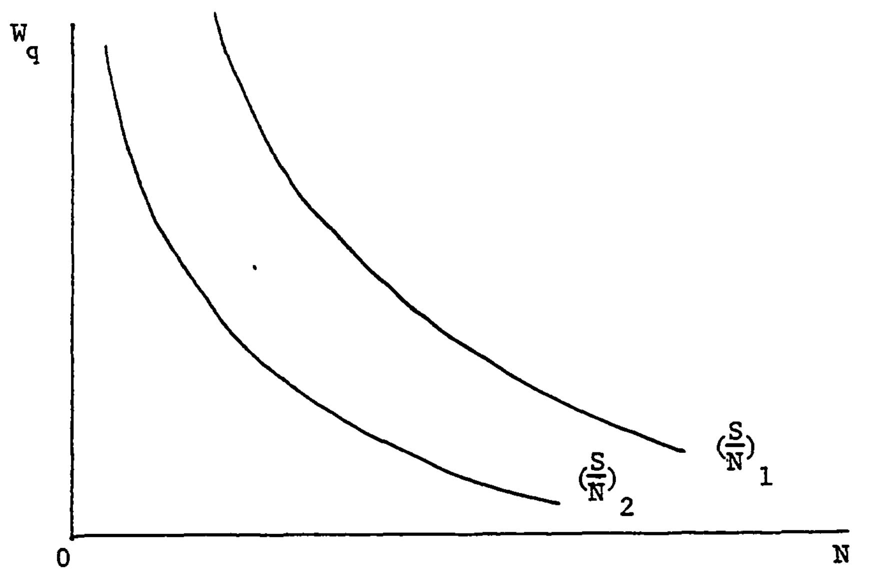

Figure 4. Expected queuing time

What Figure 4 implies is that the queuing time component of T can be reduced simply by increasing the size of the district, S/N constant. However, increasing district size will also increase travel distance to the median individuals' residences (please see Appendix B). This additional travel distance translates into additional time, as described below.

The simplest relationship to postulate between travel time (TT) and distance (D) is TT = D/V where V is the average velocity of the fire trucks. However, a more sophisticated relationship has been developed by Kolesar _et al_. [22] and tested in New York City. Their model attempts to account for the time needed for the trucks to accelerate as well as for the effects of deceleration when the trucks leave the main roads as the destination is approached. With this in mind, the following equations were constructed:

$$TT = 2(D/a)^{1/2} \text{ if } D \leq 2d, \text{ and}$$

$$TT = Vc/a + D/Vc \text{ if } D > 2d$$

where a is the rate of acceleration, Vc is the cruising velocity of the vehicles, and d is the distance required to achieve cruising velocity. Since the model as presented contains three different units of measure, it is necessary to convert all travel times into minutes as follows:

$$TT = 2 \, \frac{60D}{a}^{1/2} \text{ if } D \leq 2d, \quad \text{and}$$

$$TT = Vc/a + \frac{60D}{Vc} \text{ if } D > 2d \text{ .}$$

Estimates of the model's parameters were obtained by Kolesar _et al_. [22] for New York City. The rate of acceleration was found to be 29.0 miles/hour/minute, cruising velocity was 39.2 miles/hour, and the distance required to achieve cruising velocity was .44 miles. They also found that travel times were virtually the same at night as during the day and that the peak rush hour increased travel time by a maximum of 20%.

With this travel time model and the method of relating median travel distance to population presented in Appendix B, it is possible to establish a relationship between travel time to the median individuals' residences and district population. Since the area of a district grows more rapidly than median distance, this relationship is of the form shown in Figure 5.

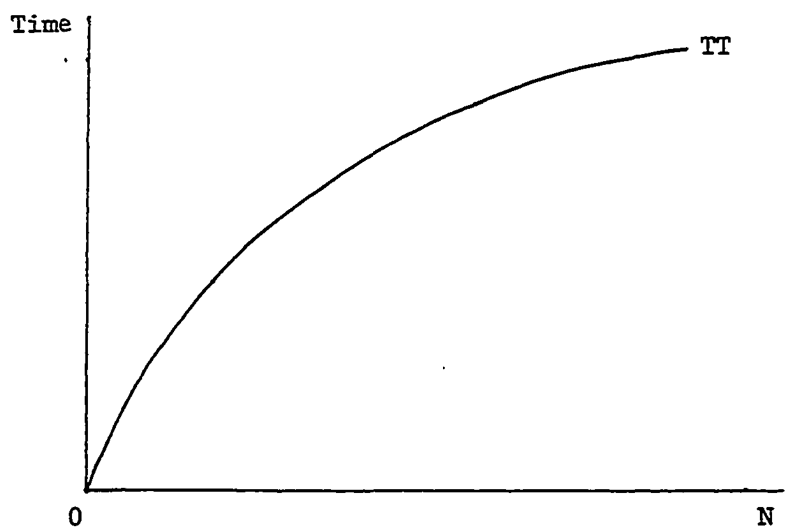

Figure 5. Expected travel time to the median individuals

In other words, TT increases as N increases, but at a decreasing rate.

Let expected response time to the median individuals' residences (RT) be defined as the sum of W_q and TT. By so doing, it is possible to determine the district size that minimizes RT for a given S/N. The idea is demonstrated in Figure 6.

N^* represents the equilibrium population size for a district with a given S/N, since it minimizes the expected response time to the median

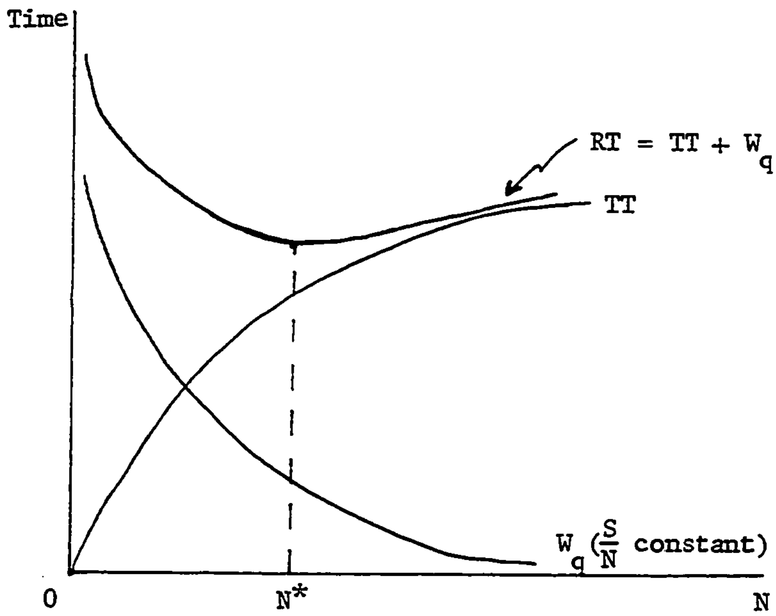

Figure 6. Minimizing RT

individuals' residences. With S/N given, determining N^* also determines S^*. Returning to Figure 1 in Chapter II, one can plot S^* and N^* as in Figure 7.

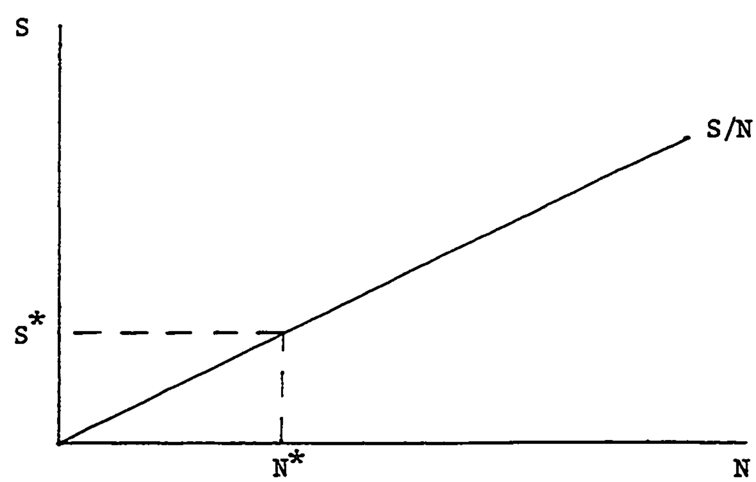

Figure 7. Plotting S^* and N^*

Note, however, that S^* can only take on integer values. With the makeup of fire companies exogenously determined, one can have one fire company or two fire companies, but not one and a half fire companies. As a result, N^* can take on only values which are integer multiples of the N in the given S/N. Suppose, for example, that S/N = 1/15,000. If S^* equals 2, N^* must equal 30,000. If S^* equals 3, N^* must equal 45,000. S^* can never equal 2.5, so N^* can never equal 37,500. This phenomenon is important to keep in mind when considering the comparative statics of the model (see below), since it imparts a certain "lumpiness" to the model.

This model can be used to predict how changes in various parameters will affect the equilibrium district size for each level of per-person cost. For example, an increase in the frequency of alarms (due either to an increase in the number of fires or in the number of false alarms) will cause the fire companies to be busy more often. This will cause the queuing function to shift out, increasing N^* as illustrated in Figure 8. Intuitively, as queuing time becomes a larger component of response time, the reductions in W_q due to increases in N (S/N constant) outweigh the increases in TT over a larger range of N. Therefore, the minimum point on the response time function is shifted rightward, and the equilibrium size of the district increases. A decrease in the frequency of alarms has the opposite effect. The complexity of the queuing time equation (see Appendix A) precludes an analytic proof of these results, but they consistently occur in simulations of the model as reported in Appendix D. The simulations also illustrate the "integers problem" alluded to above:

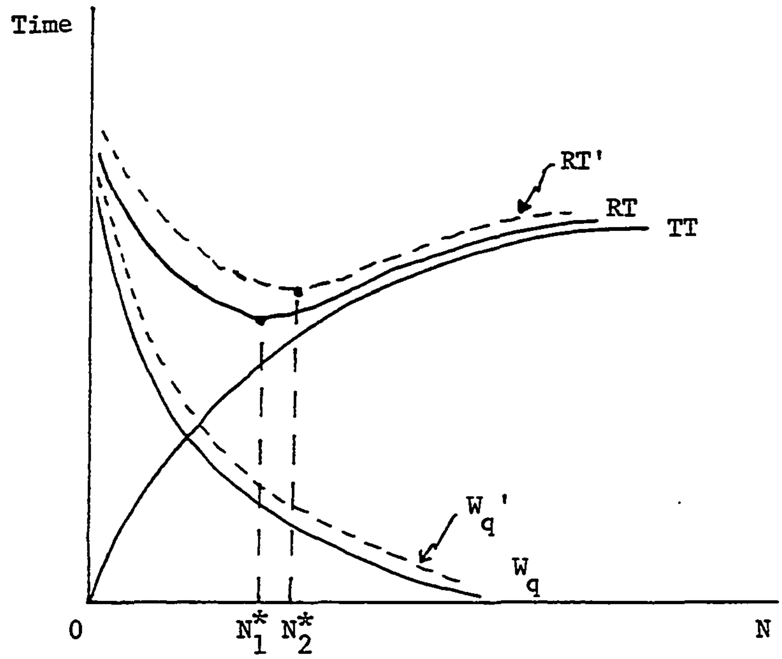

Figure 8. An increase in alarm frequencies

Since S^* and N^* can change only by discrete amounts, under some circumstances, a change in alarm frequencies will not be large enough to alter S^* or N^*. There is, in essence, a "threshold effect", which indicates that minor changes in the parameters will not always alter the location of the equilibrium.

A decrease in service time reduces the time that a unit is busy and hence shifts the queuing function down. N^* will therefore tend to decrease. Analogously, an increase in service time will increase N^*. These conclusions are likewise supported by the simulations in Appendix D and are subject to the same threshold effect.

An increase in population density decreases travel time to the Nth person's residence. All else equal, this increases N^*, since reductions in W_q will be larger than increases in TT over a larger range of N. However, less time spent traveling means that units will be available for new assignments sooner, thereby reducing queuing times. This effect tends to decrease N^*. The net effect on N^* depends upon the relative sizes of the counteracting tendencies. The impact on travel time of a change in population density is:

$$\frac{\partial TT}{\partial D} \cdot \frac{\partial D}{\partial A}$$

where A represents density. The impact on queuing time is

$$\frac{\partial W_q}{\partial \mu_B} \; \frac{\partial \mu_B}{\partial TT} \; \frac{\partial TT}{\partial D} \; \frac{\partial D}{\partial A}$$

where μ_B is the mean blocking rate.[1] Once again, however, the expression for W_q is intractable, so $\partial W_q/\partial \mu_B$ cannot be obtained. The simulations indicate that in general, an increase in population density tends to increase N^* for a given S/N, but again the threshold effect is operative.

The Expansion Path

As before, calculating S^* and N^* for each S/N allows one to define an expansion path. For club goods in general, the shape of the expansion

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[1] Blocking time is the interval between the moment one customer begins service and the moment the next customer can be served. In the case of queues in a grocery store, blocking time and service time (see Appendix A) are the same. However, in situations such as fire service, blocking time is equal to service time plus travel time between fires. This paper follows Alec Lee's advice that when appropriate, queuing models should use blocking time instead of service time [23, p. 13]. With respect to the question at hand, changes in travel time directly affect blocking time and therefore, queuing time. A failure to distinguish between service time and blocking time would be to overlook this point.

path is indeterminate. Adams [1, 2] has shown that although goods per person increases along the expansion path, the size of the club (in terms of S^* and N^*) may either rise or fall as S/N increases. However, the simulations presented in Appendix D indicate that for fire suppression services, S^* and N^* both decrease as S/N increases, as shown in Figure 9. The stepwise shape of the expansion path results from the aforementioned integers problem. The reduction of district size as the quality of service increases is intuitively reasonable, because as S/N increases, queuing time becomes less and less important. Hence, the reductions in

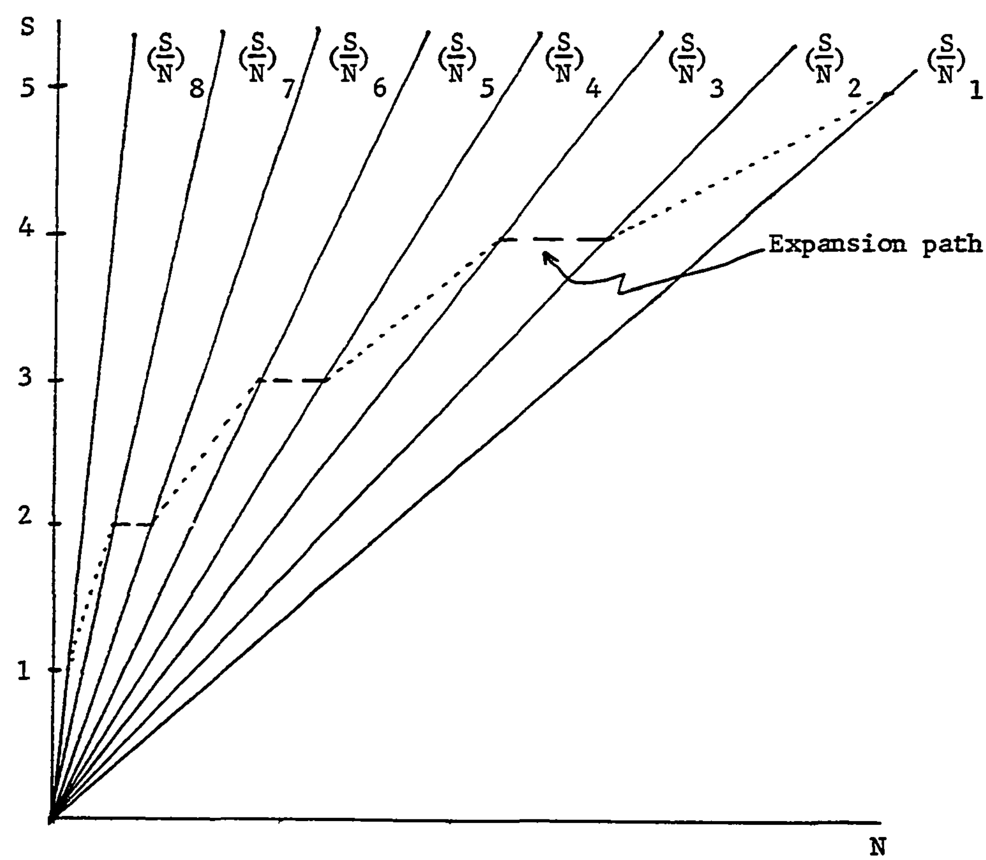

Figure 9. The expansion path

expected queuing time from increasing S and N proportionately will be offset more quickly by the increases in travel time caused by increasing N.

The same factors that were previously shown to affect the location of N^* for each S/N value affect the location of the entire expansion path in exactly the same fashion. For example, an increase in the frequency of alarms increases N^* and S^* for each S/N value and causes the expansion path to shift "up" toward larger-sized districts.

Choosing the Equilibrium S/N

Given the club configurations available as presented by the expansion path, each voter will desire the S/N which maximizes his expected utility. Recall that in this chapter utility is defined as being equal to wealth, i.e. $U = Z$. Hence, $E(U) = E(Z)$. Wealth net of dues is again designated by Z_N, and is the value of wealth if there is no fire. In the event that there is a fire, wealth will be equal to $Z_N - L$, where $L = gT$ as defined previously. With P_F being the probability of a fire occurring, expected utility is:

$$E(U) = (1 - P_F)\ (Z_N) + P_F(Z_N - gT)\ .$$

Higher levels of S/N increase expected utility by reducing expected response time, but decrease expected utility by increasing dues and therefore, reducing Z_N. For any particular individual, the utility maximizing S/N occurs where the gains from decreasing T are just offset by the losses associated with higher dues. The utility maximizing S/N of the individual with the median preference will be the equilibrium S/N, designated $(S/N)^*$. This individual need not live the median distance away from the center of the equilibrium district. However, for simplicity, the simulations assume that he does.

There are several factors that will affect the location of $(S/N)^*$. An increase in the cost of fire companies, for example, increases dues for each S/N. All else equal, this will lead to a lower $(S/N)^*$ because the reduced expected losses from having a higher S/N are more quickly outweighed by the extra expense. Note, however, that the location of the expansion path itself is unaffected if the price of fire companies is independent of the quantity a district purchases [1, 2]. Again, simulation results support this conclusion and shed light on some other points as well. For example, as the frequency of alarms declines, the choice of $(S/N)^*$ becomes more sensitive to changes in the cost of fire companies. As standard economic theory predicts, the less necessary a good becomes, the more elastic will be the demand for that good. The simulations also show a mild trend supporting the idea that a technological improvement that reduces service time will reduce $(S/N)^*$. This would occur for two reasons: First, a reduction in service time reduces queuing time, shifting the expansion path "down". Response time for each S/N will decline, which means that a lower S/N is needed to maintain the same service level. Secondly, the service time component of T declines, reducing loss in the event that there is a fire. Lower response times would, therefore, not be quite as important as they were previously. The simulations also indicate that, all else equal, a decrease in population density will tend to increase $(S/N)^*$.

With a risk neutral utility function, the marginal utility of wealth is constant. Therefore, the effect of an increase in wealth will manifest itself only in an increase in the slope of the loss function. (This is assuming that wealthier individuals have both more expensive and more numerous possessions. Hence, the wealthier an individual is, the more damage the fire will do in a given amount of time.) A reduction in expected response time is therefore worth more, and $(S/N)^*$ will increase as wealth increases. This result is also supported by our simulations.

The universal adoption of smoke or heat sensors would reduce fire detection and reporting time. This would reduce the expected value of T, and hence, reduce expected loss in the event of a fire. All else equal, this would tend to reduce $(S/N)^*$.

Fire Insurance and Rent Gradients

Until now, the possibility of purchasing fire insurance has been ignored. For risk neutral individuals, this is of no concern, since they wouldn't buy insurance even if it were available. This is because insurance companies need to charge a premium in excess of the expected value of loss in order to cover operating expenses and to make a profit. The expected value of loss is $P_F(gT)$, so the premium (R) must be greater than $P_F(gT)$. The expected utility of a risk neutral individual without insurance is

$$E(U) = (1 - P_F)Z_N + P_F(Z_N - gT) .$$

The expected utility of a risk neutral individual with insurance that promises to pay the value of the loss in the event of a fire is

$$E(U) = (1 - P_F)Z_N + P_F(Z_N - gT) + P_F(gT) - R .$$

Since it has already been established that $R > P_F(gT)$, the individual is clearly better off without the insurance.

In the general model where risk aversion is possible, fire insurance becomes a legitimate concern. In that situation, insurance becomes somewhat of a substitute for fire suppression services. It is probable that individuals will desire both insurance and fire suppression services up to the point where the ratios of their marginal expected utility to their marginal cost are equal. In addition, individuals who, due to the voting rules, do not get their utility maximizing S/N (or S and N given S/N) may use insurance as a means of adjusting. People who get worse fire protection than they want may buy more insurance than they might have with better protection, and vice versa. In addition, it should be pointed out that insurance companies link their insurance rates to the quality of fire suppression services in an area. Thus, individuals have an extra incentive to demand a higher S/N since it will also reduce their insurance rates.

It is also worth noting that with equal dues and a homogeneous population (having the same wealth and utility functions), a rent gradient will probably be established. Since response time to the edge of the district exceeds response time nearer to the center, expected loss is greater at the edge. All else equal, individuals will be better off living nearer to the center, so rents will be bid up there until rent differentials reflect the advantages of living nearer to the fire station.

CHAPTER IV. MAN-MADE RECREATIONAL FACILITIES

This chapter will extend the analysis of the preceding chapters to consider its application to man-made recreational facilities such as tennis courts. While the situations are similar in many respects, there are two significant differences between the emergency service and the recreational facilities problems. The first is that in the former case, the service is brought to the individual, while in the latter situation, the individual travels to the facility. Secondly (and more importantly), it was assumed that everyone had a potential need for a fire suppression service (i.e., everyone had a positive probability of having a fire). However, some individuals may never play tennis. Any model dealing with the provision of recreational facilities must therefore take this diversity of preferences into account.

In this chapter, the same decision-making framework as was used in the prior chapters will be employed. Let S be the number of tennis courts and N be the number of people in a district. Then the choice of S and N given S/N will be such that the utility of the persons living the median distance from the courts will be greater than utility of individuals living the median distance from the courts in any other sized district. Given the district configuration for each S/N, the individual with the median preference will ultimately decide on a particular S/N.

The Utility Function

The utility function used in this chapter will be of the form

$$U = U(Y, H, G)$$

where Y represents expenditures on all goods except those associated with H, which stands for the number of sessions the individual spends playing tennis. G represents all remaining leisure time. The individual earns income by working at a job which pays w dollars per hour.

To make the problem manageable, it will be assumed that the utility function is risk neutral with respect to the relevant variables. As will become apparent later on, these variables include the "price" of H, and hence, reference to an indirect utility function is necessary to determine risk neutrality. Appendix E discusses this process and is included for the reader's convenience. As before, the assumption of risk neutrality allows us to focus exclusively on the expected values of all variables.

The Elements of the Problem

As before, there is a one-to-one correspondence between per-person cost and the ratio of courts to people (S/N). Let P be the (annualized) cost of each tennis court. Per-person cost (dues) can then be defined as (PS)/N. If P is independent of S, dues are linearly related to S/N. It will be useful later on to have dues expressed on a daily basis, so let us define $P_S = P/365$, so dues will be equal to

$$M = \frac{P_S S}{N} .$$

This "tennis tax" is a lump-sum tax that must be paid whether or not the individual uses the tennis courts. If the courts <u>are</u> used, then the individual incurs certain costs per trip, both in terms of time and money. Time costs include queuing time, round-trip travel time, and the time needed to play a session of tennis. Let the expected value of these variables be designated as W_q, TT, and X, respectively. Monetary cost is the round-trip transportation cost equal to eT_D where e is the cost per mile and T_D is the distance in miles to and from the courts.

If we monetize the time costs by multiplying them by w, the opportunity cost of time, we can determine the expected value of the cost of one trip to the tennis courts. Let us designate this as the expected price of good H (P_H), and

$$P_H = (W_q + TT + X)w + eT_D \ .$$

The value of X is completely up to the individual, and is assumed to be independent of district size. However, the configuration of the district will influence the values of W_q, TT, and T_D.

Choosing S and N for Each Level of Per-Person Cost

As noted above, a given level of per-person cost (dues) implies a given S/N. To determine the equilibrium sized district for a particular S/N, it is necessary to select S and N so that P_H is minimized for those individuals living the median distance from the center of the district. Once again, W_q declines as S and N increase, S/N constant, but T_D and TT for the median individuals increase as N increases. P_H is therefore minimized for persons living the median distance from the center when S and N are such that $(W_q + TT)w + eT_D$ is minimized, as pictured in Figure 10. (Recall that X is independent of S and N.) TT w may be less than or greater than eT_D, depending on the values of w and e. The values of TT and T_D displayed on the graph are those for the individual who would live the median distance from the courts.

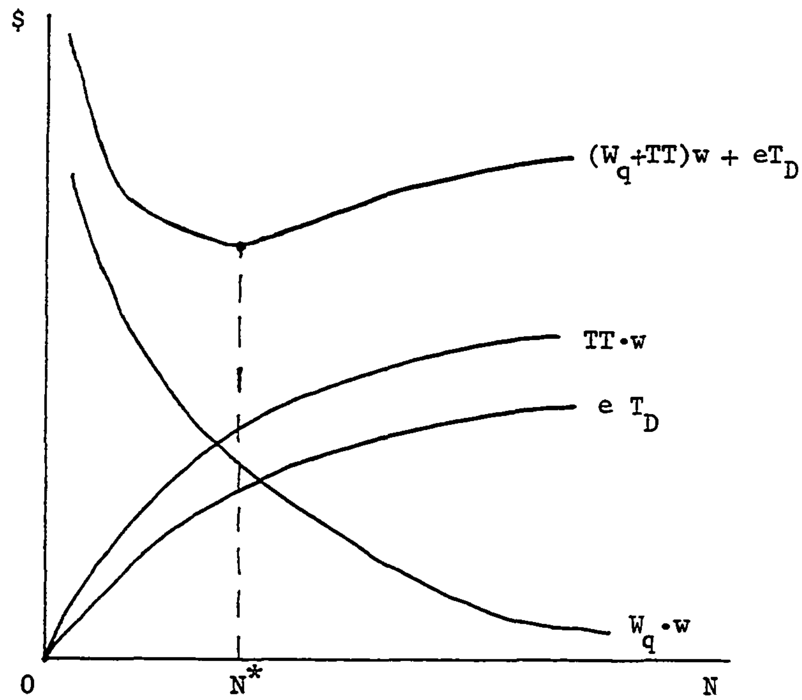

Figure 10. Minimizing P_H for the median person

Given S/N, determining N^* automatically determines S^*, as was demonstrated in Figure 7.

While no simulations have been run on this model, it is likely that the comparative statistics of it are similar to those of the risk neutral fire suppression model. For example, a general increase in the frequency that people play tennis will cause $W_q \cdot w$ to shift up, increasing N^*. An increase in X would have the same effect, as the longer people stay on the courts, the longer others will have to wait for a court to become available, all else equal.

Changes in e and w will also affect the location of N^* and S^*. If the price of gasoline, for example, were to increase, eT_D would shift up. This would make travel distance more significant than previously, probably reducing N^*. An increase in w will shift both $TT \cdot w$ and $W_q \cdot w$ up, with the net effect being ambiguous without resorting to simulations.

Utility Maximization and the Equilibrium S/N

As before, an expansion path can be obtained by determining S^* and N^* for each level of S/N. Given these configurations, each individual will desire the S/N that maximizes his utility.

For a particular S/N, dues and P_H will be known. The utility level for a given S/N can, therefore, be determined by maximizing

$$U = U(Y, H, G)$$

subject to the constraint that

$$24w = P_y Y + [(W_q + TT + X)w + eT_D]H + wG + M$$

or

$$24w = P_y Y + P_H H + wG + M .$$

This constraint simply says that the individual's expenditures of money and (monetized) expenditures of time cannot exceed their maximum. 24w represents the individual's maximum potential income. P_y is the price of "good" Y. Since Y is essentially an income variable, from now on P_y will be defined as being equal to one. As discussed previously, P_H is the cost per trip to the tennis courts. G is multiplied by w so as to reflect the opportunity cost of leisure. M represents the mandatory lump sum tennis tax (dues) associated with the particular S/N.

We are now in a position to set up a Lagrangian, L, in order to solve the utility maximizing problem:

$$L = U(Y, H, G) + \lambda(24w - Y - P_H H - wG - M) .$$

The first order conditions for utility maximization are:

$$\partial L/\partial Y = \partial U/\partial Y - \lambda = 0$$
$$\partial L/\partial H = \partial U/\partial H - \lambda P_H = 0$$
$$\partial L/\partial G = \partial U/\partial G - \lambda w = 0$$
$$\partial L/\partial \lambda = 24w - Y - P_H H - wG - M .$$

The usual conditions for a utility maximum hold here as well. The ratios of any two marginal utilities are equal to the price ratios of the two goods in question. What makes this problem unusual is that by choosing a different S/N, the individual is able to alter the constraint in his utility maximizing problem. A different S/N means both a different M and a different P_H, since W_q, TT, and T_D will change. The exposition of this point can be facilitated by means of graphs. (The problem cannot be solved by fully differentiating the first order conditions because P_H contains W_q as an element. As noted previously, W_q is undifferentiable.)

Consider a particular S/N with its associated M and P_H. The individual's budget plane would be as depicted in Figure 11.

The intercept on each axis is obtained by solving the constraint in the utility maximization problem with other variables equal to zero. For example, to obtain the intercept on the G axis, set Y = H = 0 and solve for G:

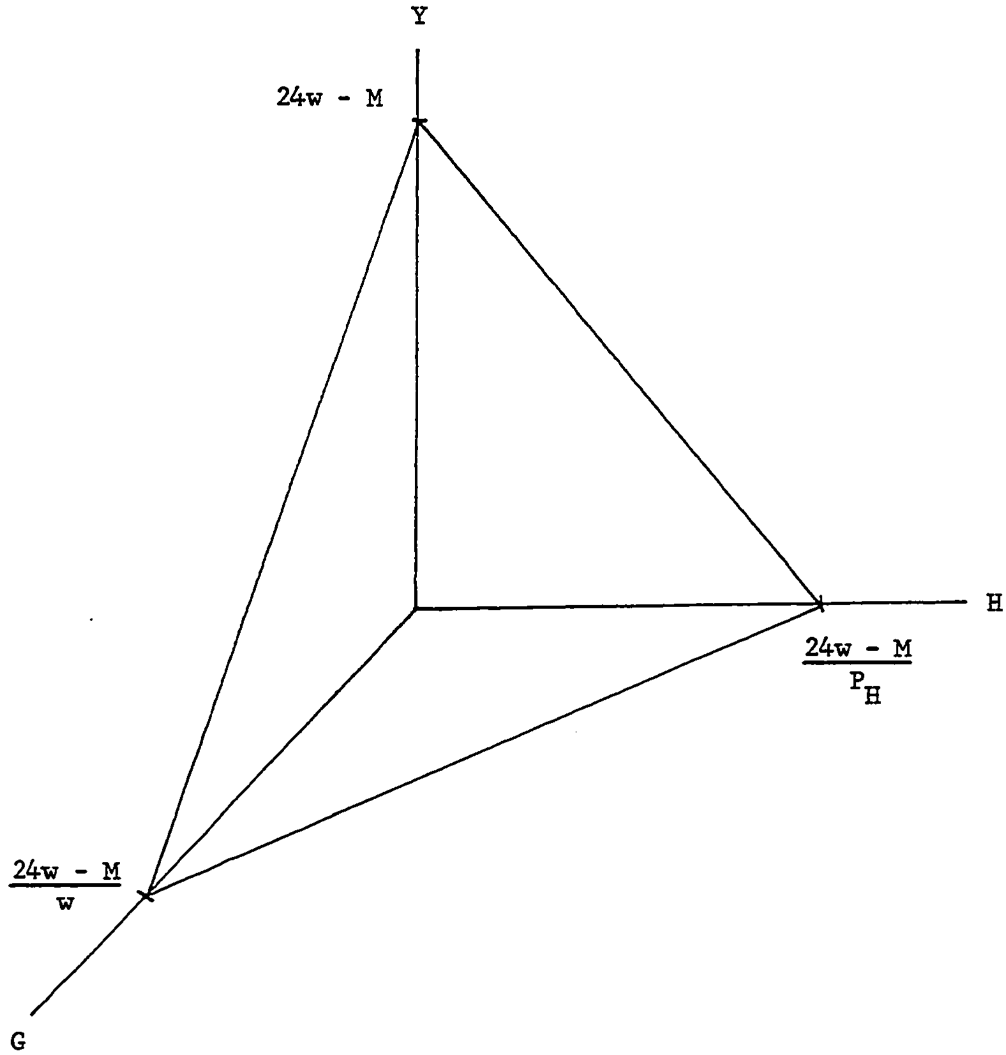

Figure 11. The budget plane

$$24w = Y + P_H H + wG + M$$

$$24w = wG + M$$

or

$$G = \frac{24w - M}{w} \quad .$$

Let us now consider the effect of increasing (S/N) to (S/N)'. First of all, this will cause dues to rise from M to M'. The intercepts on the Y and G axes will unequivocally move in towards the origin. It will also tend to cause the intercept on the H axis to shift in, but the increase in S/N will cause W_q, TT, and T_D to decline[1], decreasing P_H. Thus, the net effect on the fraction

$$\frac{24w - M}{P_H}$$

is indeterminate. However, as we shall see, the only relevant case is when the decrease in the denominator overwhelms the decrease in the numerator, causing the intercept on the H axis to shift out as pictured in Figure 12.

If the decrease in the numerator caused by the increase in M overwhelms the decrease in the denominator caused by the decline in P_H (via the decrease in W_q, TT, and T_D), then the new budget plane will lie everywhere inside of the old budget plane, clearly making the individual worse off. This can be said without reference to a utility function because the increase in dues is greater than the savings in terms of the

[1]Provided we are moving along the expansion path.

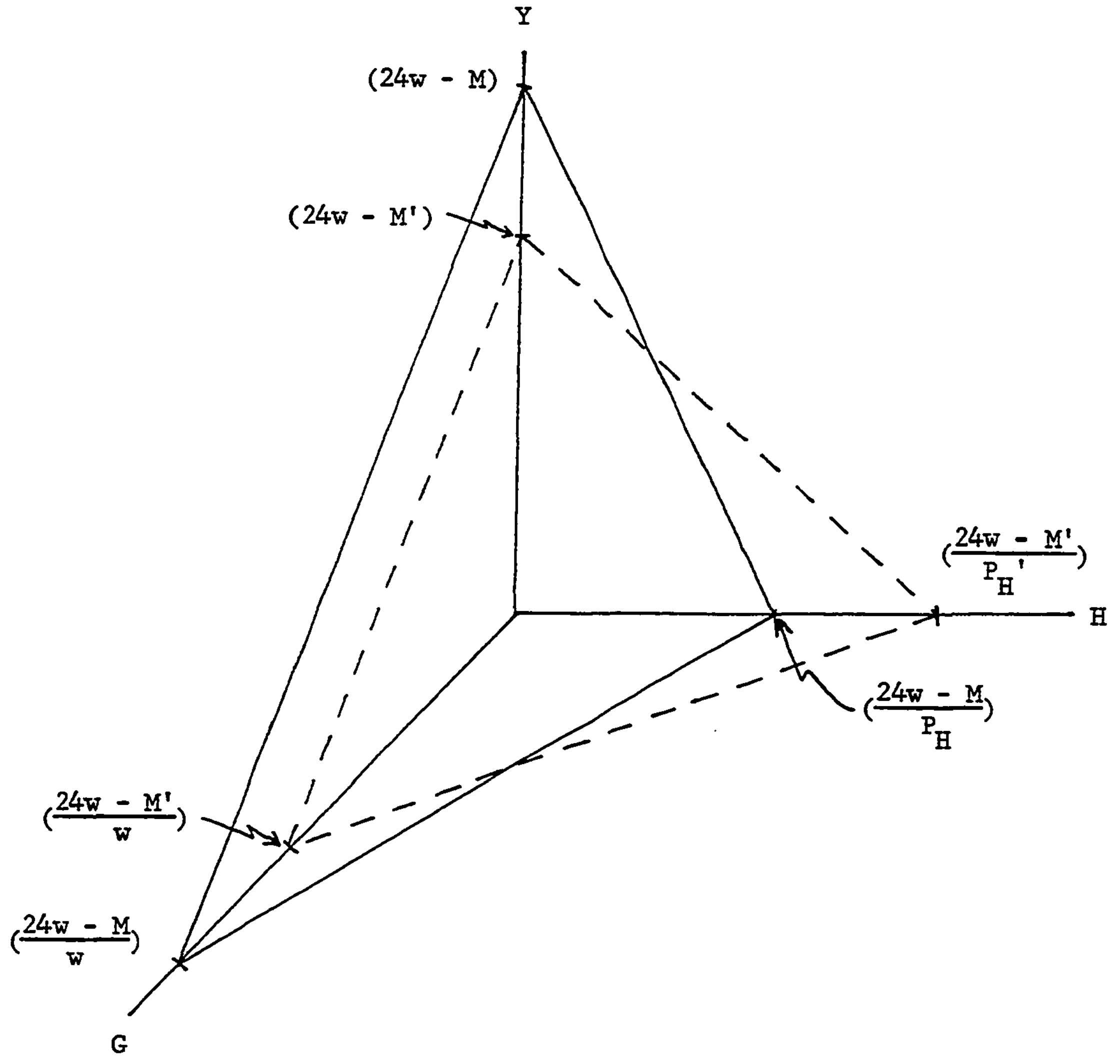

Figure 12. Shifting the budget plane

lower cost of traveling to the tennis court even if the individual made the maximum number of trips to the court possible. This situation is likely to occur when S/N is already relatively large. In such a case, P_H is relatively small, and any reductions in P_H would also be small. This reduction would be less than the increase in dues, $\partial M/\partial(S/N) = P_S$. The situation is graphically represented in Figure 13.

Returning now to the case where increasing S/N causes the H intercept to shift out, the question of utility maximization still remains. Indifference surfaces in Y-H-G space can be obtained by taking the total differential of the utility function and setting it equal to zero as follows:

$$U = U(Y, H, G)$$

so

$$dU = \frac{\partial U}{\partial Y} dY + \frac{\partial U}{\partial H} dH + \frac{\partial U}{\partial G} dG = 0 \ .$$

The location of these indifference surfaces are fixed. Thus, changing S/N shifts the budget plane, but not the indifference surfaces. The utility maximizing individual will therefore choose the S/N that is associated with the budget plane that allows him or her to reach the highest possible indifference surface. The utility maximum may be a point such as U^* in Figure 14.

Selecting a different S/N means shifting the budget plane and obtaining a tangency on a different indifference surface. If U^* is the utility maximum, then all other values of S/N result in budget planes tangent to lower and indifference surfaces. Mathematically, solving for U^* involves finding the point where

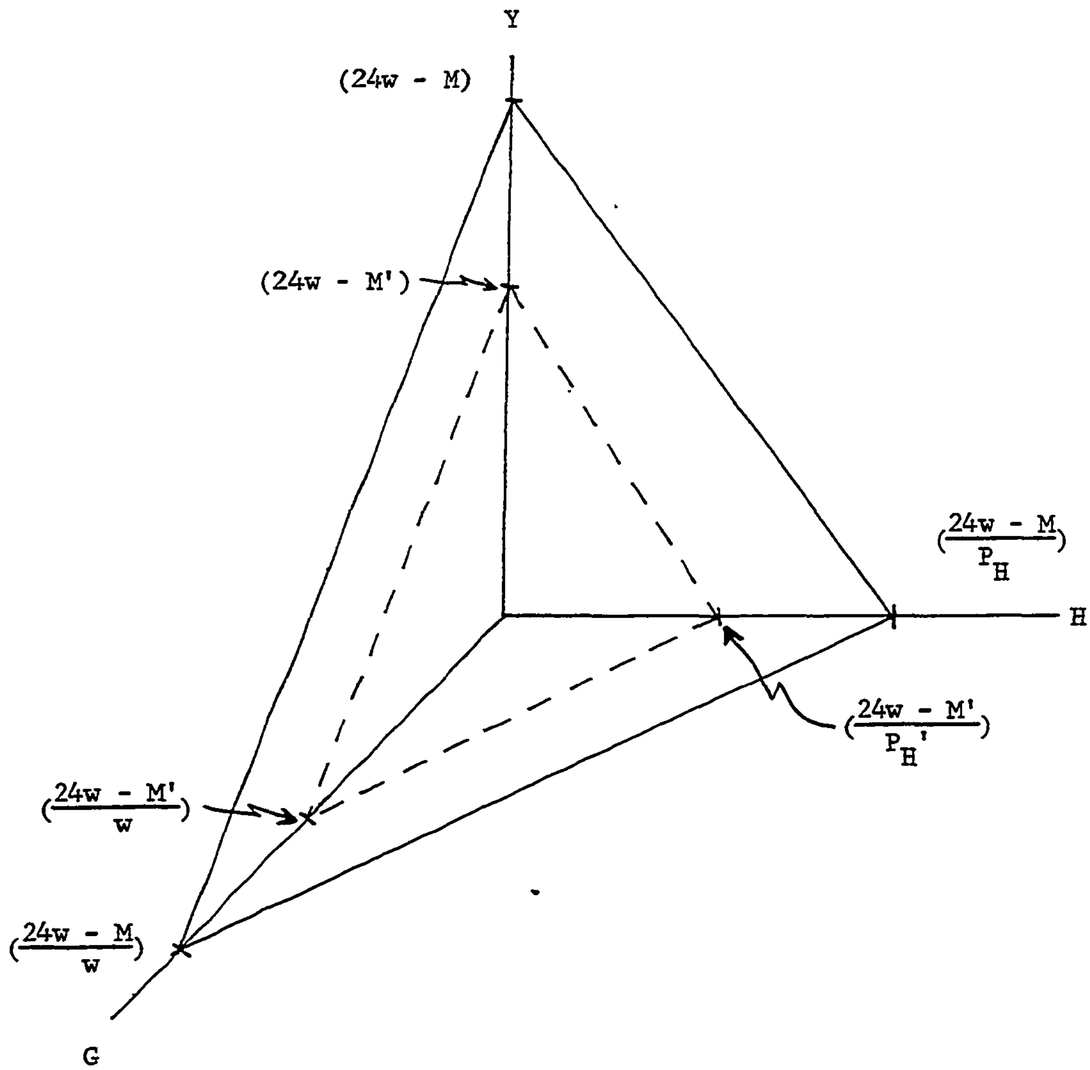

Figure 13. An inferior new budget plane

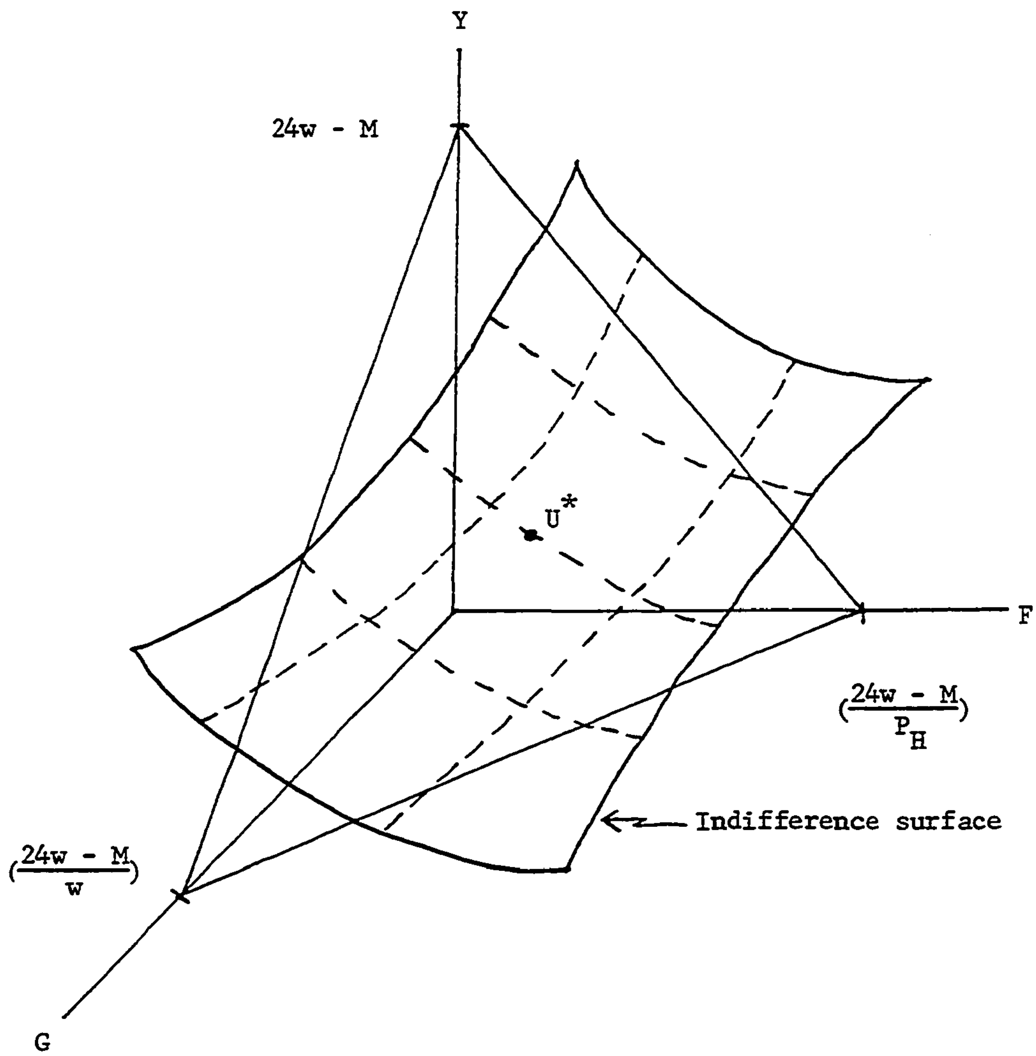

Figure 14. Utility maximization

$$\frac{dU}{d(S/N)} = \frac{\partial U}{\partial Y}\frac{\partial Y}{\partial(S/N)} + \frac{\partial U}{\partial H}\frac{\partial H}{\partial(S/N)} + \frac{\partial U}{\partial G}\frac{\partial G}{\partial(S/N)} = 0 .$$

Once again, no analytic solution to this problem is possible because the queuing time equation is intractable.

It is completely possible that, for non-tennis players in particular, the utility maximizing S/N = 0. In other words, some people would prefer having no tennis courts and therefore, no dues, as in Figure 15.

It should also be noted that the utility maximizing S/N can be different for two individuals having identical utility functions. This is because they may live in different locations, and hence, TT and T_D will differ. Thus, the equilibrium sized district for any particular S/N will result in different budget planes for each person. Consider, for example, a particular S/N with its associated equilibrium S^* and N^*. Individual A lives close to the center of an equilibrium district, while individual B lives near the edge of the district. Although A and B are identical in all other respects (not only having identical utility functions, but also the same X, e and w), TT and T_D will be larger for B than for A. Thus, P_H will be larger for B than for A, which is reflected in different budget planes as illustrated in Figure 16. With A and B having different budget planes, it is not likely that they will have the same utility maximizing S/N.

Returning to the larger story, each person will vote for the S/N which maximizes his utility, and the voter with the median preference will dominate. This final result determines the number of "tennis districts" to create, the number of people in each district, and the number of

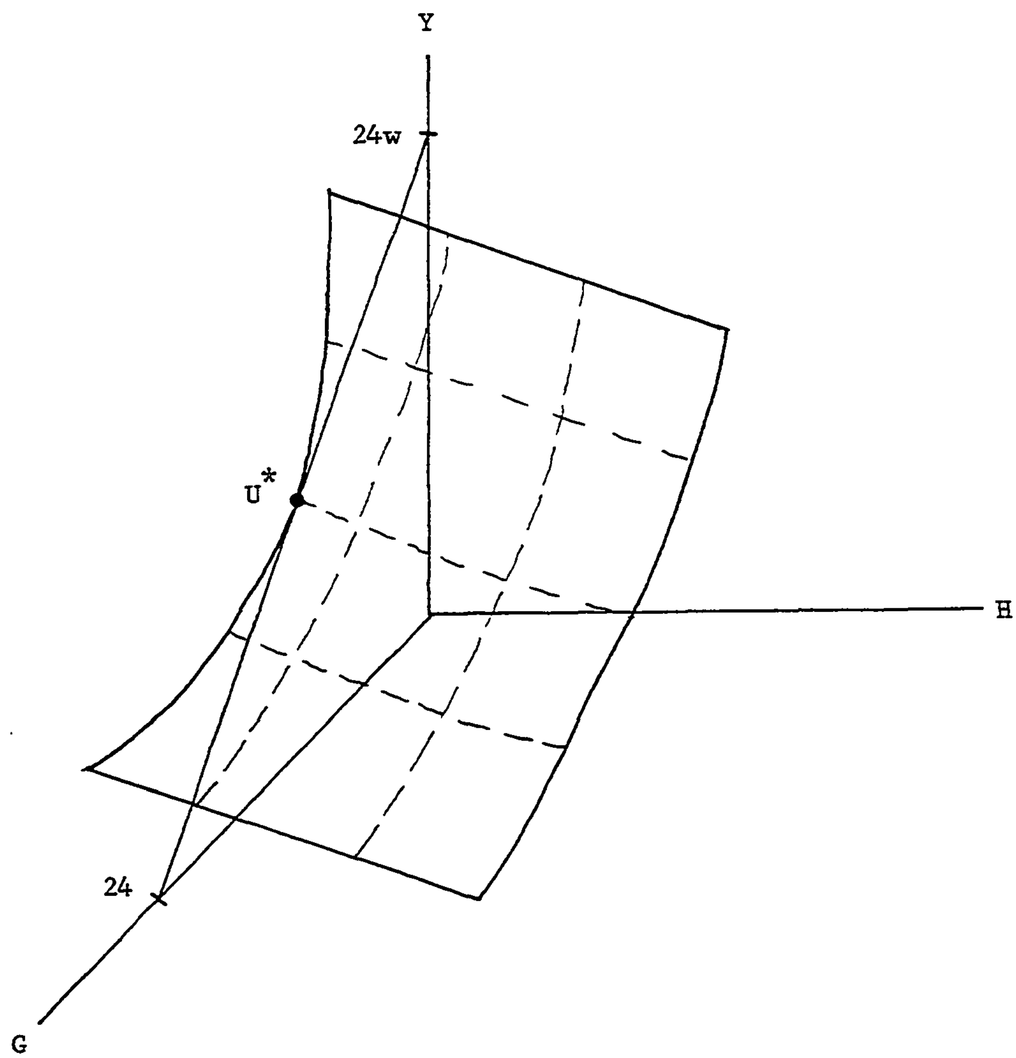

Figure 15. U^* for a non-tennis player

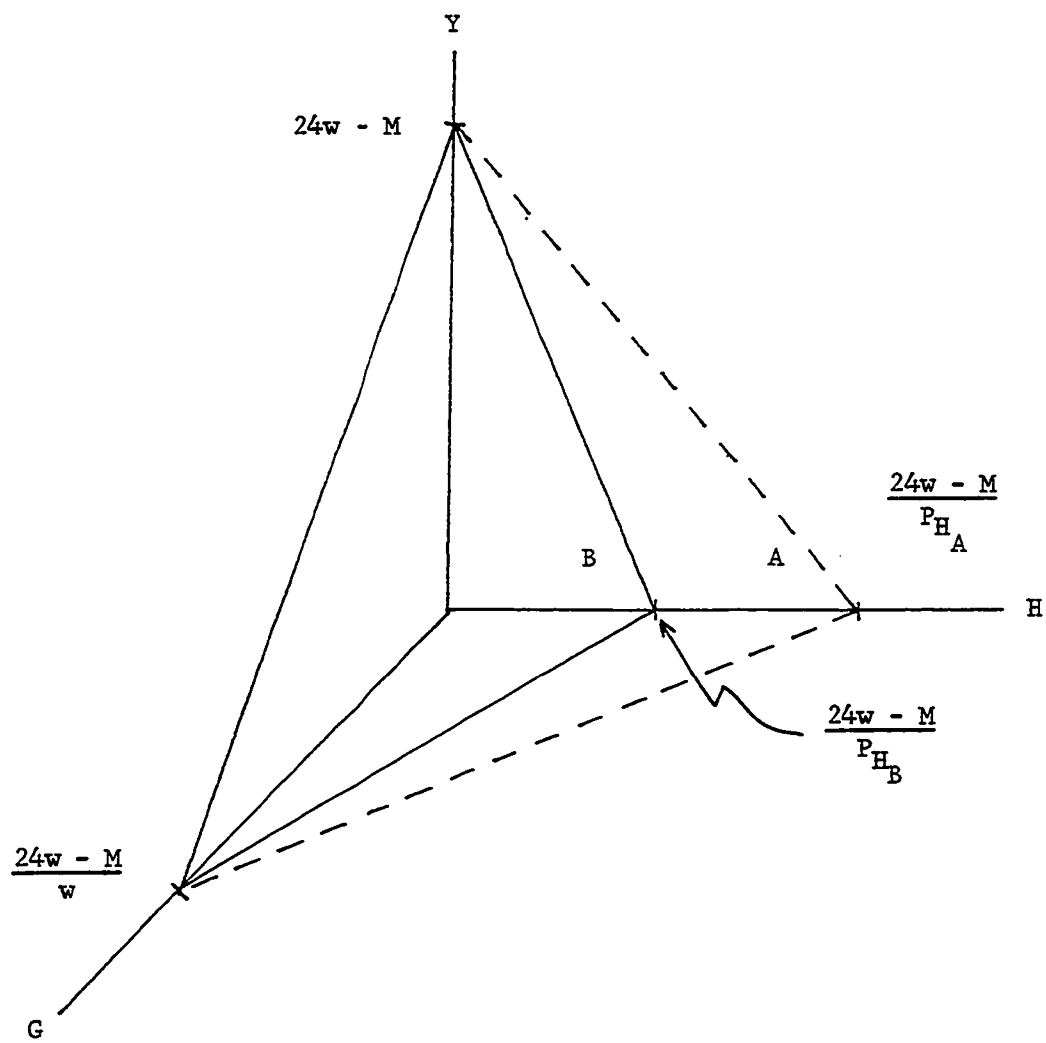

Figure 16. Locational differences illustrated

courts to build in each district. Suppose, for example, that the equilibrium S/N, $(S/N)^*$, is 1/20,000, and is associated with $S^* = 2$ and $N^* = 40,000$. If the city's population is 200,000 people, there will be 5 districts (200,000/40,000), with 2 courts and 40,000 people in each district.

CHAPTER V. CONCLUSION

This dissertation has explained why and given examples of how spatial and temporal considerations might be integrated into the economic theory of clubs. Space and time are important elements in determining the quality of club service, and a failure to incorporate them has been a serious deficiency in formal club theory.

A model of fire suppression districts has been constructed which shows how equilibrium districts are determined under a particular decision-making framework. Items that affect the location of the equilibrium have also been discussed. Simulations of the restricted model were presented which supported the arguments in the text. Space, which requires time to move through, and time variables are central to the model. This is because time is a crucial variable when an emergency such as a fire occurs.

In the tennis court example, there are no longer any emergencies, but the model reflects the fact that time itself is a scarce resource. As such, time has an opportunity cost which must be considered. Space enters the model because it requires the expenditure of both time and other resources (reflected by monetary costs) to move through.

The models used in this dissertation are not offered as definitive efforts, but rather as prototypes upon which further analysis can be based. Possible extensions include incorporating more location theory, employing more sophisticated operations research techniques, allowing for different financing schemes and altering the decision-making framework. The models might also serve as the basis of empirical work.

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ACKNOWLEDGMENTS

It would be impossible to list all of the people who have helped me complete this dissertation and this degree. Yet certain people deserve special recognition and thanks.

First among these is my major professor, Roy Adams, whose performance has been far beyond the call of duty. His door has always been open, and he has selflessly devoted huge quantities of his time and energy to helping me find my way through the labyrinth. I could not have asked for a better teacher or friend.

The other members of my committee also deserve my thanks and appreciation: Roy Hickman, who was the first person to make statistics comprehensible to me; Dudley Luckett, who inspired a great interest in the history of economic thought in me; Charles Meyer, who introduced me to public finance and was always willing to share his wisdom on a wide variety of topics; and James Prescott, who, among other things, provided me with an interesting and extremely eventful summer research project.

Thanks also go to Jerry Post for his miracle working at the computer, and to Arnold Faden, Roy Gardner, Harvey Lapan, Mark McNulty, Jim Paulsen, and Dan Royalty for their mathematical and statistical expertise.

Tom Kompas deserves special recognition for generating a genuine enthusiasm for economics in me.

My years in Ames have been blessed with a wealth of friends. Therefore, I will risk the wrath of many by listing a few of them here: Keith Heimforth, Tom Boggess, Susie Hagen, Fusako Higashide, Pat Hurst, Mathew Mathai, Masoud Moghaddam, Andy Morton, Steven Steyn, Robert Thomas, Don Umatum, Lynnette Walsh, and Kathy Way.

Despite what has recently transpired, a very special thank you goes to Susan Reinhart. You were there when I needed you.

And finally, a profound thank you to AMORC.

APPENDIX A: QUEUING THEORY

Introduction

Queuing theory is the application of mathematical and statistical techniques to problems associated with "queues" or waiting lines. It was originally developed in response to the needs of business and industry, where queuing problems are quite common.

Consider, for example, the predicament of the manager of a busy supermarket. If he hires too few clerks, the length of the check-out lines may become intolerably long, ultimately resulting in a loss of customers. On the other hand, if he hires too many clerks, he will find that he is paying them to stand around idle for most of the day. Determining the optimal number of clerks to hire is a classic queuing theory problem.

The Structure of the Queuing Process

The queuing process begins when a "customer" (not necessarily human) arrives at the queuing system and requires service. If all available servers are occupied, a line forms. Eventually, the customer is served and then leaves the system.

More formally, customers are generated from an "input source" or population. This population may be classified as either finite or infinite, though in practice any finite but "large" population may be treated as infinite [26, p. 9]. The pattern in which members of the population arrive is typically, though not always, assumed to follow a Poisson distribution. Examples of customers arriving at a queuing system include people lining up at a bank teller's window, ships arriving at a port to unload their cargo, papers being placed on a typist's desk, and phone calls for help arriving at a fire station.

If all available servers are busy when the customers arrive, a queue forms. In some cases, the length of the queue itself is limited, perhaps by the size of the "waiting room". But even when the queue has no such limitations, it is possible that the customer may choose not to enter the queuing system if the line is too long. Such behavior is referred to as "balking".

The order in which customers leave the queue to be served is known as the queue discipline. The most common type is "first-in-first-out" or FIFO discipline. Other types include "last-in-first-out" (LIFO), "served in random order" (SIRO), and variations of these which incorporate a priority procedure. An example of this last type can be found at a hospital emergency room where critically injured patients are treated ahead of less seriously ill people who have been waiting longer.

The queuing system may possess one or more servers. It is these individuals (or machines) who perform the task(s) that the customer requires. The time that it takes to accomplish this task is called the service time. If the service time is virtually identical for every customer, one may speak of a constant service time distribution. However, the length of the service time frequently varies with each customer. Hence, it becomes necessary to specify probability distributions for service times. The most common is the exponential distribution, but the Erlang (gamma) distribution is also frequently appropriate.

A Basic Model

Modeling a real-world queuing process must begin by collecting information about the system in question. One must determine the size of the population, the nature of the arrival pattern, the maximum queue length, queue discipline, the number of servers, and the distribution of service times. For the sake of illustration, let us assume that we are examining a fairly standard queuing system with the following characteristics:

1) The population is large enough that the rate of arrival may be treated as being independent of the number of people in the system.
2) The arrival pattern of new customers follows a Poisson distribution with mean λ. (λ is, therefore, the mean arrival rate per unit of time.)
3) Arrivals form a single queue which has no upper bound. There is no balking.
4) Customers are served in order of arrival. (FIFO queue discipline.)
5) There are s servers.
6) The service time distribution is exponential with mean μ. Hence, $1/\mu$ is the expected service time.
7) The utilization factor ρ is defined as $\lambda/(s\mu)$. ρ must be less than one, since if $\rho \geq 1$, the queue will become infinitely large. ($\rho > 1$ implies $\lambda > s\mu$. In such a situation, customers arrive faster than the system can process them.)

With such information, queuing theory allows one to calculate various steady-state characteristics of the system. For this particular model, the formulae needed to obtain most of the important characteristics have been derived and are readily available. Some of these are reproduced below [19, pp. 421-422].

1) The probability that there are no customers in the system is:

$$P_o = \frac{1}{\sum\limits_{N=0}^{s-1} \frac{(\lambda/\mu)^N}{N!} + \frac{(\lambda/\mu)^s}{s!} \frac{1}{1-(\lambda/(s\mu))}} \ .$$

2) The expected length of the queue is:

$$L_Q = \frac{P_o(\lambda/\mu)^s \rho}{s!(1-\rho)^2} \ .$$

3) The expected waiting time in the queue (net of service time) is:

$$W_q = \frac{L_Q}{\lambda} \ .$$

4) The expected waiting time in the system (gross of service time) is:

$$W = W_q + \frac{1}{\mu} \ .$$

5) The probability distribution of waiting times is:

$$P(W > t) = e^{-\mu t} \quad 1 + \frac{P_o(\lambda/\mu)^s}{s!(1-\rho)} \ \frac{1-e^{-\mu t(s-1-\lambda/\mu)}}{s-1-\lambda/\mu} \ .$$

When $s-1-\lambda/\mu = 0$, this becomes

$$P(W) > t) = e^{-\mu t} \quad 1 + \frac{P_o(\lambda/\mu)^s}{s!(1-\rho)} \mu t \ .$$

Similar results are available for other types of queuing models. However, there is a large group of queuing models for which either no general solutions have been found, or the solutions obtained are so intractable as to be virtually useless for practical applications. As a result, Monte Carlo simulations are frequently employed as a means of estimating the characteristics of queuing systems. Such simulations are especially important when:

1) The queue discipline is not FIFO,
2) The arrival pattern is not Poisson, or
3) The service time distribution is not constant, exponential or Erlang.

It should be emphasized that the above formulae apply to queuing systems in a steady state. This is due primarily to what Alec Lee called "the first working rule of queuing theory", namely, "time dependent solutions to queuing models are either unobtainable or unmanageable" [23, p. 26]. Nevertheless, one must recognize the fact that many queuing systems may never reach a steady state, or if they do, may not stay there long. But even in such circumstances, queuing models may provide some insight into the system. As Lee points out,

> "It is possible for an engineer to make a great deal of practical progress by using formulas for the properties of gases derived from the models of statistical mechanics (which also assume steady-state conditions). Similarly, the operational research practitioner can make much headway by using steady-state formulas" [23, pp. 215-126].

APPENDIX B: DETERMINING MEDIAN DISTANCE

Consider a square, diamond-shaped district such as the one depicted below in Figure B1.

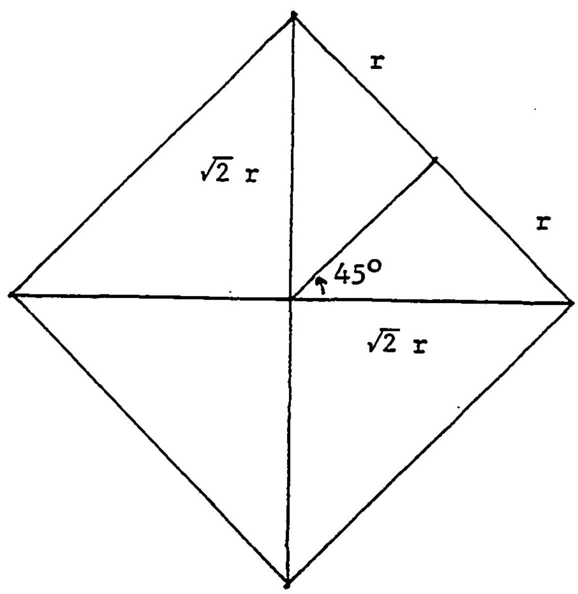

Figure B1. A fire suppression district

Distance r is the shortest distance from the center of the diamond to its perimeter. (This distance on any diamond will henceforth be called its "short radius".) Basic geometry allows us to describe the length of one side of the diamond as being equal to $r + r = 2r$. The Pythagorean theorem allows us to determine the distance from the center of the diamond to a corner as being equal to $\sqrt{2}\ r$.

With the length of one side of the diamond being equal to 2r, the area of the diamond is $(2r)^2 = 4r^2$. If population density is (uniformly) equal to A, then the total number of people in the district is equal to $A4r^2$.

Consider another diamond with Z being the short radius, and $Z < r$, as shown in Figure B2.

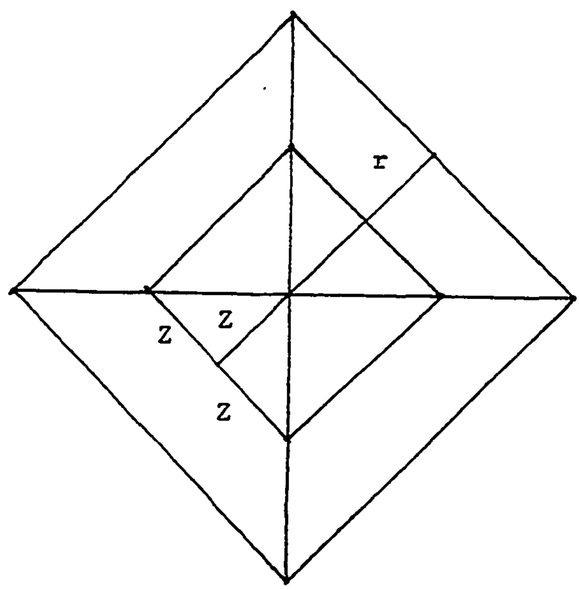

Figure B2. Concentric diamond Z

The perimeter of such a diamond is 8Z. The number of people living along the perimeter of this diamond is A8Z.

If we allow Z to vary from o to r, and then sum up the total number of people, we get

$$\int_o^r A8Z\, dZ = A4r^2$$

which is the total number of people in the original diamond, as already demonstrated.

The percentage of the total population living on the perimeter of any particular diamond is

$$\frac{A8Z}{A4r^2} = \frac{2Z}{r^2} \; .$$

$(2Z)/r^2$ is the density function of population. From it we can obtain both the mean and the median distances.

Recall that the mean of any variable X is

$$E(X) = \int Xf(X)dX$$

where f(X) is the density function of X. Thus, the mean distance is

$$E(Z) = \int_o^r Z \frac{2Z}{r^2} \, dZ = 2/3 \; r \; .$$

The median, m, is obtained by solving for m where

$$\int_o^m \frac{2Z}{r^2} \, dZ = 1/2$$

implying that $m = \frac{1}{\sqrt{2}} r$. Median distance is, therefore, greater than mean distance, since $1/\sqrt{2} > 2/3$.

Note that along a road grid, the distance to every point on the perimeter of a diamond from the center is the same. Consider, for example, the diamond which has distance m for its short radius, as shown in Figure B3.

Let the road grid run north-south and east-west. Distance m becomes distance 2e along the grid. (Note that $\sqrt{2}$ m, the distance to the far corner, is also equal to 2e along the road grid.) So:

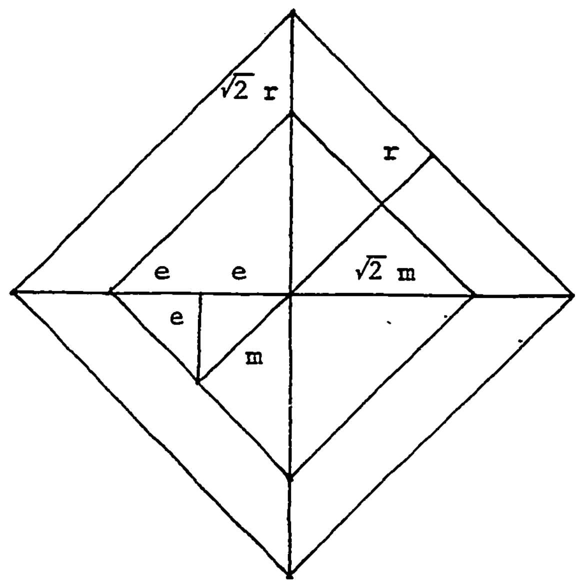

Figure B3. Distance along a road grid

$$m^2 = e^2 + e^2$$

or

$$e = \frac{m}{\sqrt{2}}$$

and

$$2e = \frac{2m}{\sqrt{2}} \; .$$

Recall that $m = \frac{1}{\sqrt{2}}\, r$, so

$$2e = \frac{2(\frac{1}{\sqrt{2}})r}{\sqrt{2}} = r \; .$$

Travel distance along a road grid to the perimeter of a diamond which contains half of the district's population is therefore equal to the short radius of the entire district.

APPENDIX C: OBTAINING THE DENSITY FUNCTION OF A RANDOM VARIABLE WHICH IS EQUAL TO THE SUM OF OTHER RANDOM VARIABLES

Let Y, X, and Z be random variables such that $Y = X + Z$. If X and Z are independent, the density function of Y may be obtained from the density functions of X and Z[1].

To illustrate the procedure, consider the following case, where X is distributed uniformly (a, b), and Z is distributed exponentially (λ). The density functions are therefore

$$f_x(X) = \frac{1}{b - a} \qquad a \leq X \leq b$$

and

$$f_z(Z) = \lambda e^{-\lambda Z} \qquad Z \geq 0 \; .$$

If X and Z are independent, then

$$f_{x,z}(X,Z) = f_x(X) \cdot f_z(Z) \; .$$

The distribution function of Y is

$$F_y(t) = P(Y \leq t)$$

and

$$P(Y \leq t) = P(X + Z \leq t) \; .$$

From this information one can obtain the density function of Y, which comes in two parts:

---

[1]The density function of Y can be obtained even if X and Z are not independent, but the process is much more complicated.

Part I: $a \leq t \leq b$

$$P(X + Z \leq t) = \int_a^t \int_o^{t-x} f_{x,z}(X,Z)dZdX$$

$$= \int_a^t \int_o^{t-x} \frac{1}{b - a} \lambda e^{-\lambda Z} dZ\, dX = \frac{1}{b - a} (t - \frac{1}{\lambda} - a + \frac{1}{\lambda} e^{-\lambda t + \lambda a}) .$$

To obtain the density function of Y where $a \leq t \leq b$, one must take the derivative of the above distribution function with respect to t, which results in the expression

$$\frac{1}{b - a} (1 - e^{-\lambda t + \lambda a}) .$$

Part II: $b < t$

$$P(X + Z \leq t) = \int_a^b \int_o^{t-x} f_{x,z}(X,Z)dZdX$$

$$= \int_a^b \int_o^{t-x} \frac{1}{b - a} \lambda e^{-\lambda Z} dZ\, dX = \frac{1}{b - a} [b - a +$$

$$\frac{1}{\lambda} (e^{-\lambda t + \lambda a} - e^{-\lambda t + \lambda b})] .$$

Taking the derivative with respect to t results in

$$\frac{1}{b - a} (e^{-\lambda t + \lambda b} - e^{-\lambda t + \lambda a}) .$$

Hence,

$$f_y(t) = \frac{1}{b - a} (1 - e^{-\lambda t + \lambda a}) \qquad a \leq t \leq b$$

$$= \frac{1}{b - a} (e^{-\lambda t + \lambda b} - e^{-\lambda t + \lambda a}) \qquad t > b .$$

It should be noted that the procedure can be extended to the more general case where

$$Y = \sum_{i=1}^{N} X_i \,.$$

With respect to obtaining the density function for T, recall that

$$T = \sum_{i=1}^{N} X_i$$

where X_1 = detection and reporting time,

X_2 = queuing time,

X_3 = travel time, and

X_4 = service time.

In the basic queuing model of Appendix A, service time is assumed to have an exponential distribution. The density function of queuing time can be derived from the fact that

$$P(W_q > t) = (1 - \sum_{N=0}^{s-1} P_N)\, e^{-s\mu(1-\rho)t}$$

where P_N is the probability that there are exactly N people in the queue at any time t [19, p. 422].

The distribution function of W_q is therefore

$$P(W_q \leq t) = 1 - P(W_q > t)$$

$$= 1 - (1 - \sum_{N=0}^{s-1} P_N)\, e^{-s\mu(1-\rho)t} \,.$$

Hence, the density function of W_q is

$$\frac{d[P(W_q \leq t)]}{dt} = s\mu(1-\rho)\,(1 - \sum_{N=0}^{s-1} P_N)\, e^{-s\mu(1-\rho)t} \qquad t \geq 0 \,.$$

The distributions of travel time and detection and reporting time might be hypothesized as having either truncated normal or uniform distributions. The exact form must, of course, be empirically verified.

Obtaining the actual density function of T would involve specifying the exact density functions of all four component variables, and then solving the necessary quadruple integrals.

APPENDIX D: SIMULATION RESULTS

The results of calculating values for S^* and $(S/N)^*$ under various situations are presented in the accompanying table. The key to that table is as follows:

K represents the population size required to generate one fire alarm per hour. Thus, the larger K is, the smaller the frequency of alarms for any given sized population. The numbers chosen for K stem from a chart presented on page 8 of Chaiken and Larson [6]. In 1968, at peak time on the highest alarm day, there were 111 alarms per hour in New York City. The population of New York City in 1970 was 7,894,851 [36]. Dividing that population by 111 produces the number 71,125, representing K at peak time on a peak day. According to the same chart, peak time on an average day generated 48 alarms per hour, resulting in K equal to 164,476. K equal to 263,162 represents peak time on the lowest alarm day of the year, when 30 alarms per hour were reported.

U_s represents mean service rate, i.e. the number of fires which a fire company can extinguish per hour. $U_s = 2$ implies service time averages 30 minutes per fire, $U_s = 3$ implies a 20-minute average, and so on. Note that service time here includes both "set-up time" and "roll-up time" (i.e. the time needed to set-up the fire hoses, etc., as well as the time needed to roll them up again).

"A" represents population density. In 1970, New York City had a population density of 26,345 people per square mile. The number 39,517 was chosen because it is 50% greater than 26,345, or just slightly larger than the density of the Bronx (35,721 people per square mile in 1970).

Table D1. Simulation results

K	U_s	A	Values of S^* S/N $\frac{1}{40,000}$	$\frac{1}{35,000}$	$\frac{1}{30,000}$	$\frac{1}{25,000}$	$\frac{1}{20,000}$	$\frac{1}{15,000}$	$\frac{1}{10,000}$
71,125	2	39,517	5	5	4	4	3	3	2
		26,345	5	5	4	4	3	3	2
		13,172	5	4	4	3	3	3	2
		2,634	4	4	3	3	3	2	2
	3	39,517	4	3	3	3	3	2	2
		26,345	4	3	3	3	3	2	2
		13,172	3	3	3	3	2	2	2
		2,634	3	3	3	2	2	2	2
	4	39,517	3	3	3	2	2	2	2
		26,345	3	3	3	2	2	2	2
		13,172	3	3	2	2	2	2	2
		2,634	3	2	2	2	2	2	2
164,476	2	39,517	3	3	3	2	2	2	2
		26,345	3	3	3	2	2	2	2
		13,172	3	3	2	2	2	2	2
		2,634	2	2	2	2	2	2	2
	3	39,517	2	2	2	2	2	2	2
		26,345	2	2	2	2	2	2	2
		13,172	2	2	2	2	2	2	2
		2,634	2	2	2	2	2	2	1
	4	39,517	2	2	2	2	2	2	2
		26,345	2	2	2	2	2	2	2
		13,172	2	2	2	2	2	2	1
		2,634	2	2	2	2	2	1	1
263,162	2	39,517	2	2	2	2	2	2	2
		26,345	2	2	2	2	2	2	2
		13,172	2	2	2	2	2	2	2
		2,634	2	2	2	2	2	2	2

	Values of (S/N)* (N is in thousands)			Values of (S/N)* (N is in thousands)		
		C			WT WB BR	
1 / 5,000	$500,000	$1,000,000	$1,500,000	$50,000 37,500 45,000	$100,000 75,000 90,000	$150,000 112,500 135,000
2	1/5	1/10	1/15	1/15	1/10	1/5
2	1/5	1/10	1/10	1/15	1/10	1/5
2	1/5	1/10	1/10	1/10	1/10	1/5
2	1/5	1/5	1/5	1/10	1/5	1/5
2	1/5	1/10	1/10	1/15	1/10	1/10
2	1/5	1/10	1/10	1/15	1/10	1/10
2	1/5	1/10	1/10	1/15	1/10	1/5
2	1/5	1/5	1/10	1/10	1/5	1/5
2	1/5	1/10	1/15	1/15	1/10	1/10
2	1/5	1/10	1/15	1/15	1/10	1/10
2	1/5	1/10	1/10	1/15	1/10	1/5
1	1/5	1/10	1/10	1/10	1/10	1/5
2	1/10	1/15	1/20	1/30	1/15	1/15
2	1/10	1/15	1/20	1/25	1/15	1/15
2	1/10	1/15	1/20	1/20	1/15	1/10
2	1/5	1/10	1/15	1/15	1/10	1/10
2	1/15	1/20	1/25	1/30	1/20	1/15
1	1/10	1/20	1/25	1/30	1/20	1/15
1	1/10	1/15	1/20	1/25	1/15	1/15
1	1/5	1/10	1/15	1/15	1/10	1/5
1	1/15	1/20	1/25	1/30	1/20	1/15
1	1/10	1/20	1/25	1/30	1/20	1/15
1	1/10	1/15	1/20	1/25	1/15	1/15
1	1/5	1/10	1/10	1/15	1/10	1/10
2	1/15	1/25	1/30	1/35	1/25	1/20
2	1/15	1/25	1/30	1/35	1/25	1/20
2	1/15	1/20	1/25	1/30	1/20	1/15
1	1/10	1/15	1/20	1/20	1/15	1/10

Table D1. (Continued)

			Values of S^*						
							S/N	S/N	
K	U_s	A	$\frac{1}{40,000}$	$\frac{1}{35,000}$	$\frac{1}{30,000}$	$\frac{1}{25,000}$	$\frac{1}{20,000}$	$\frac{1}{15,000}$	$\frac{1}{10,000}$
	3	39,517	2	2	2	2	2	2	2
		26,345	2	2	2	2	2	2	2
		13,172	2	2	2	2	2	2	1
		2,634	2	2	2	2	1	1	1
	4	39,517	2	2	2	2	2	1	1
		26,345	2	2	2	2	2	1	1
		13,172	2	2	2	2	2	1	1
		2,634	2	1	1	1	1	1	1

	Values of (S/N)* (N is in thousands)			Values of (S/N)* (N is in thousands)		
		C			WT WB BR	
1/5,000	$500,000	$1,000,000	$1,500,000	$50,000 37,500 45,000	$100,000 75,000 90,000	$150,000 112,500 135,000
1	1/20	1/30	1/35	1/45	1/30	1/20
1	1/15	1/25	1/35	1/40	1/25	1/20
1	1/15	1/25	1/30	1/35	1/25	1/20
1	1/10	1/15	1/15	1/20	1/15	1/10
1	1/20	1/30	1/40	1/50	1/30	1/25
1	1/15	1/30	1/35	1/45	1/30	1/20
1	1/15	1/25	1/30	1/35	1/25	1/15
1	1/10	1/15	1/15	1/20	1/15	1/10
	(Wealth held constant) WT = $100,000 WB = 75,000 BR = 90,000			(Cost held constant) C = $1,000,000		

The number 13,172 is one-half of 26,345, and 2634 is only 10% of 26,345, or roughly equal to the population density of Ames, Iowa (2,321 people per square mile in 1970).

C represents the cost of maintaining one fire company for one year. Chaiken and Larson [6, p. 10] indicate that the cost of operating a single fire engine now frequently exceeds $500,000 per year, and many fire companies typically consist of two fire engines.

WT represents the total wealth of the median individual. WB represents the portion of that wealth which would be jeopardized by a fire. BR is equivalent to the letter "g" in the text and is the rate at which fire will destroy wealth per hour. The values for BR are "guestimated" from page 23 of reference 20, where Ignall <u>et</u> <u>al</u>. present a ballpark figure of $1000 per minute of damage on a WB equal to $50,000. Hence, WB = $75,000 implies BR = $1,500 per minute or $90,000 per hour, etc.

The table reveals several things. First, an expansion path is obtained by holding K, U_s, and A constant while changing S/N. Larger values of S/N tend to result in smaller values of S^* (and therefore, smaller N^*). In some cases, S^* does not change due to the integers problem; nevertheless, a constant S^* does result in smaller N^* as S/N increases. To illustrate the point, consider the case where K = 71,125, $U_s = 2$, and A = 39,517. For S/N = 1/40,000, $S^* = 5$, implying an $N^* = 200{,}000$. For S/N = 1/10,000, $S^* = 2$, implying an $N^* = 20{,}000$. For S/N = 1/5,000, S^* still equals 2, but N^* will not only equal 10,000.

Also, for a given S/N, S^* tends to decline as either K or U_s increases or as A decreases. An increase in K or U_s means that queuing time declines, so travel time becomes relatively more important. A decrease in A also means that travel time begins to dominate queuing considerations sooner, making for smaller districts in terms of both S and N. Again, however, the integers phenomenon continues with changes being discrete rather than continuous.

As discussed in the text, the choice of the expected utility maximizing point along an expansion path $[(S/N)^*]$ is influenced by such things as cost, wealth, and the rate at which fire losses occur. The section of the table which is the second from the right examines the effect of changes in C under various conditions, holding WT, WB and BR constant. Keeping the integers problem in mind, an increase in C causes $(S/N)^*$ to fall, illustrating the law of demand. For example, for $K = 164{,}476$, $U_s = 2$ and $A = 39{,}517$, $(S/N)^*$ goes from 1/10,000 to 1/15,000 to 1/20,000 as C rises from $500,000 to $1,000,000 to $1,500,000. Furthermore, the effect seems to be more pronounced as K increases, implying that demand is more elastic as the frequency of alarms declines. Compare, for example, the case cited above to the situation where $K = 263{,}162$, keeping U_s and A constant. Here, $(S/N)^*$ falls from 1/15,000 to 1/25,000 as C rises from $500,000 to $1,000,000, indicating a greater sensitivity to cost. This result should be expected, since with lower alarm frequencies, the need for fire services is reduced, and it is well-known that the less necessary an item is, the more elastic will be the demand for that item.

In addition, for any given C, $(S/N)^*$ tends to decline as K, U_s, and/or A increase. The integers problem blurs this result, as an increase in these parameters will not always reduce $(S/N)^*$. The effect of U_s is less pronounced than the effects of K and A, and there is one example of an increase in U_s actually increasing $(S/N)^*$ (U_s changing from 2 to 3 for C = \$1,500,000, K = 71,125, and A = 39,517). Nevertheless, these trends exist throughout the table, which the reader can verify for his/herself.

In the right-hand section of the table, the effects of changes in wealth (for a constant level of C) are illustrated. Assuming that WT, WB, and BR change in proportion to one another, an increase in WT does, as expected, tend to cause $(S/N)^*$ to increase. For example, if K = 164,476, U_s = 3, and A = 39,517, $(S/N)^*$ rises from 1/30,000 to 1/20,000 to 1/15,000 as WT rises from \$50,000 to \$100,000 to \$150,000. As with cost changes, the "wealth effect" seems to be more pronounced as K increases, implying a greater wealth elasticity as the frequency of alarms falls. To illustrate, compare the above situation to the case where K = 263,162. Here, as wealth increases over the same range, $(S/N)^*$ increases from 1/45,000 to 1/30,000 to 1/20,000.

Finally, for any given WT, an increase in K, U_s, or A causes $(S/N)^*$ to fall. Once again, the integers problem seems to cloud the issue, but the trend is definitely observable.

APPENDIX E: RISK NEUTRALITY AND INDIRECT UTILITY FUNCTIONS

It is sometimes necessary to determine if a utility function is risk neutral (or risk averse) in terms of a parameter which is not directly in the utility function. Such is the case in the text where utility is a function of non-tennis expenditures, tennis playing, and leisure, it is necessary to test for neutrality with respect to the "price" of tennis playing. In this situation, one must convert the (direct) utility function into an indirect utility function, which defines utility in terms of income and prices rather than in terms of goods, services, and leisure. If the indirect utility function is linear with respect to the parameter in question, then the direct utility function is risk neutral with respect to that parameter.

To illustrate the idea, let us consider a utility function of the form

$$U = X^a Y^b Z^c$$

where $a + b + c = 1$ and X, Y, and Z are private goods with prices P_x, P_y, and P_z. Suppose that the individual has income I, and one wants to know whether he or she is risk-neutral with respect to that income.

The first step is to derive the demand curves for X, Y, and Z from the first order conditions for utility maximization. Mathematically, maximize $U = X^a Y^b Z^c$ subject to the constraint that $I = P_x X + P_y Y + P_z Z$. The Lagrangian is

$$L = X^a Y^b Z^c + \lambda[I - P_x X - P_y Y - P_z Z]$$

and the first order conditions are

$$\frac{\partial L}{\partial X} = aX^{a-1} Y^b Z^c - \lambda P_x = 0$$

$$\frac{\partial L}{\partial Y} = bX^a Y^{b-1} Z^c - \lambda P_y = 0$$

$$\frac{\partial L}{\partial Z} = cX^a Y^b Z^{c-1} - \lambda P_z = 0$$

$$\frac{\partial L}{\partial \lambda} = I - P_x X - P_y Y - P_z Z = 0 \ .$$

Solving these equations for X, Y, and Z gives one the demand curves:

$$X^* = \frac{aI}{P_x}$$

$$Y^* = \frac{bI}{P_y}$$

$$Z^* = \frac{cI}{P_z} \ .$$

Substituting these values back into the utility function gives one the utility maximum, given prices and income:

$$U = (X^*)^a \ (Y^*)^b \ (Z^*)^c \ .$$

Substitution in the values of X^*, Y^*, and Z^* gives one the indirect utility function, V, in terms of prices and income:

$$V = \left(\frac{aI}{P_x}\right)^a \left(\frac{bI}{P_y}\right)^b \left(\frac{cI}{P_z}\right)^c$$

$$V = I^{(a+b+c)} \left(\frac{a}{P_x}\right)^a \left(\frac{b}{P_y}\right)^b \left(\frac{c}{P_z}\right)^c \ .$$

Recall that $a + b + c = 1$, and that P_x, P_y, and P_z are constants. Thus,

$$V = I \cdot k$$

where

$$k = \left(\frac{a}{P_x}\right)^a \left(\frac{b}{P_y}\right)^b \left(\frac{c}{P_z}\right)^c$$

and k is a constant. Indirect utility is therefore a linear function of income, so the original utility function is said to be risk-neutral with respect to income. Note, however, that this particular utility function is not risk neutral in the prices of the goods, since the indirect utility function is not linear in prices.

With respect to the problem in the text, it might be sufficient to treat the problem as if individuals were risk neutral with respect to the price of playing tennis since it is typically such a small fraction of total expenditures.